The thing which made this a bit difficult was that the range of the n-indices here was outside $[0, 2^m N - 1]$ (which describe the legal indices in the basis V_m), so that we had to use the periodicity of ϕ.

Exercise 5.16: Direct sums

Let $C_1, C_2 \ldots, C_n$ be independent vector spaces, and let $T_i : C_i \to C_i$ be linear transformations. The direct sum of T_1, T_2,. . . ,T_n, written as $T_1 \oplus T_2 \oplus \ldots \oplus T_n$, denotes the linear transformation from $C_1 \oplus C_2 \oplus \cdots \oplus C_n$ to itself defined by

$$T_1 \oplus T_2 \oplus \ldots \oplus T_n(\boldsymbol{c}_1 + \boldsymbol{c}_2 + \cdots + \boldsymbol{c}_n) = T_1(\boldsymbol{c}_1) + T_2(\boldsymbol{c}_2) + \cdots + T_n(\boldsymbol{c}_n)$$

when $\boldsymbol{c}_1 \in C_1$, $\boldsymbol{c}_2 \in C_2$, $\ldots$, $\boldsymbol{c}_n \in C_n$. Also, when $A_1, A_2, \ldots, A_n$ are square matrices, $\mathrm{diag}(A_1, A_2, \cdots, A_n)$ is defined as the block matrix where the blocks along the diagonal are $A_1, A_2, \ldots, A_n$, and all other blocks are 0. Show that, if $\mathcal{B}_i$ is a basis for C_i then

$$[T_1 \oplus T_2 \oplus \ldots \oplus T_n]_{(\mathcal{B}_1,\mathcal{B}_2,\ldots,\mathcal{B}_n)} = \mathrm{diag}([T_1]_{\mathcal{B}_1}, [T_2]_{\mathcal{B}_2}, \ldots, [T_n]_{\mathcal{B}_n}).$$

Solution. If $\boldsymbol{b}_j$ is the j'th basis vectors in $\mathcal{B}_i$, column j in $[T_i]_{\mathcal{B}_i}$ is $[T_i(\boldsymbol{b}_j)]_{\mathcal{B}_i}$. Since $(T_1 \oplus T_2 \oplus \ldots \oplus T_n)(\boldsymbol{b}_j) = T_i(\boldsymbol{b}_j)$, the column in $[T_1 \oplus T_2 \oplus \ldots \oplus T_n]_{(\mathcal{B}_1,\mathcal{B}_2,\ldots,\mathcal{B}_n)}$ corresponding to the basis vector $\boldsymbol{b}_j$ in $\mathcal{B}_i$ is

$$(0, ..., 0, [T_i(\boldsymbol{b}_j)]_{\mathcal{B}_i}, 0, ..., 0)^T,$$

which equals the corresponding column in $\mathrm{diag}([T_1]_{\mathcal{B}_1}, [T_2]_{\mathcal{B}_2}, \ldots, [T_n]_{\mathcal{B}_n})$.

Exercise 5.17: Eigenvectors of block diagonal matrices

Assume that T_1 and T_2 are square matrices, and that the eigenvalues of T_1 are equal to those of T_2. What are the eigenvalues of $\mathrm{diag}(T_1, T_2)$? Can you express the eigenvectors of $\mathrm{diag}(T_1, T_2)$ in terms of those of T_1 and T_2?

Solution. Assume that λ is an eigenvalue common to both T_1 and T_2. Then there exists a vector $\boldsymbol{v}_1$ so that $T_1\boldsymbol{v}_1 = \lambda\boldsymbol{v}_1$, and a vector $\boldsymbol{v}_2$ so that $T_2\boldsymbol{v}_2 = \lambda\boldsymbol{v}_2$. We now have that

$$\mathrm{diag}(T_1, T_2)\begin{pmatrix}\boldsymbol{v}_1\\ \boldsymbol{v}_2\end{pmatrix} = \begin{pmatrix}T_1 & 0\\ 0 & T_2\end{pmatrix}\begin{pmatrix}\boldsymbol{v}_1\\ \boldsymbol{v}_2\end{pmatrix} = \begin{pmatrix}T_1\boldsymbol{v}_1\\ T_2\boldsymbol{v}_2\end{pmatrix} = \begin{pmatrix}\lambda\boldsymbol{v}_1\\ \lambda\boldsymbol{v}_2\end{pmatrix} = \lambda\begin{pmatrix}\boldsymbol{v}_1\\ \boldsymbol{v}_2\end{pmatrix}.$$

This shows that λ is an eigenvalue for $\mathrm{diag}(T_1, T_2)$ also, with $\begin{pmatrix}\boldsymbol{v}_1\\ \boldsymbol{v}_2\end{pmatrix}$ an eigenvector.

Exercise 5.18: Invertibility of block diagonal matrices

Assume that A and B are square matrices which are invertible. Show that $\text{diag}(A, B)$ is invertible, and that $(\text{diag}(A, B))^{-1} = \text{diag}(A^{-1}, B^{-1})$.

Solution. We have that

$$\begin{aligned}\text{diag}(A, B)\text{diag}(A^{-1}, B^{-1}) &= \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} \\ &= \begin{pmatrix} AA^{-1} & 0 \\ 0 & BB^{-1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = I\end{aligned}$$

where we have multiplied as block matrices. This proves that $\text{diag}(A, B)$ is invertible, with inverse $\text{diag}(A^{-1}, B^{-1})$.

Exercise 5.19: Multiplying block diagonal matrices

Let A, B, C, D be square matrices of the same dimensions. Show that

$$\text{diag}(A, B)\text{diag}(C, D) = \text{diag}(AC, BD).$$

Solution. We have that

$$\text{diag}(A, B)\text{diag}(C, D) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} = \begin{pmatrix} AC & 0 \\ 0 & BD \end{pmatrix} = \text{diag}(AC, BD)$$

where we again have multiplied as block matrices.

Exercise 5.20: Finding N

Assume that you run an m-level DWT on a vector of length r. What value of N does this correspond to? Note that an m-level DWT performs a change of coordinates from $\boldsymbol{\phi}_m$ to $(\boldsymbol{\phi}_0, \boldsymbol{\psi}_0, \boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_{m-2}, \boldsymbol{\psi}_{m-1})$.

Exercise 5.21: Different DWTs for similar vectors

In Figure 5.1 we have plotted the DWT's of two vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. In both vectors we have 16 ones followed by 16 zeros, and this pattern repeats cyclically so that the length of both vectors is 256. The only difference is that the second vector is obtained by delaying the first vector with one element.

You see that the two DWT's are very different: For the first vector we see that there is much detail present (the second part of the plot), while for the second vector there is no detail present. Attempt to explain why this is the case. Based on your answer, also attempt to explain what can happen if you change the point of discontinuity for the piecewise constant function in the left part of Figure 5.11 in the book to something else.