Fourier series: basic concepts

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Square-integrable functions, Definition 1.6

The set of continuous, real functions defined on an interval $[0, T]$ is denoted $C[0, T]$.

A real function f defined on $[0, T]$ is said to be square integrable if f^2 is Riemann-integrable, i.e., if the Riemann integral of f^2 on $[0, T]$ exists,

$$\int_0^T f(t)^2 \, dt < \infty.$$

The set of all square integrable functions on $[0, T]$ is denoted $L^2[0, T]$.

Inner product spaces, Theorem 1.7

Both $L^2[0, T]$ and $C[0, T]$ are vector spaces. Moreover, if the two functions f and g lie in $L^2[0, T]$ (or in $C[0, T]$), then the product fg is Riemann-integrable (or in $C[0, T]$). Moreover, both spaces are inner product spaces with inner product defined by

$$\langle f, g \rangle = \frac{1}{T} \int_0^T f(t)g(t)\, dt,$$

and associated norm

$$\|f\| = \sqrt{\frac{1}{T} \int_0^T f(t)^2 dt}.$$

Fourier series, Definition 1.8

Let $V_{N,T}$ be the subspace of $C[0,T]$ spanned by the set of functions given by

$$\mathcal{D}_{N,T} = \{1, \cos(2\pi t/T), \cos(2\pi 2t/T), \cdots, \cos(2\pi Nt/T), \\ \sin(2\pi t/T), \sin(2\pi 2t/T), \cdots, \sin(2\pi Nt/T)\}.$$

The space $V_{N,T}$ is called the *N'th order Fourier space*. The Nth-order Fourier series approximation of f, denoted f_N, is defined as the best approximation of f from $V_{N,T}$ with respect to the inner product defined by (1.3).

Fourier coefficients, Theorem 1.9

The set $\mathcal{D}_{N,T}$ is an orthogonal basis for $V_{N,T}$. In particular, the dimension of $V_{N,T}$ is $2N+1$, and if f is a function in $L^2[0,T]$, we denote by $a_0,\ldots,a_N$ and $b_1,\ldots,b_N$ the coordinates of f_N in the basis $\mathcal{D}_{N,T}$, i.e.

$$f_N(t) = a_0 + \sum_{n=1}^{N} \left(a_n \cos(2\pi nt/T) + b_n \sin(2\pi nt/T)\right).$$

The $a_0,\ldots,a_N$ and $b_1,\ldots,b_N$ are called the (real) Fourier coefficients of f, and they are given by

$$a_0 = \langle f, 1\rangle = \frac{1}{T}\int_0^T f(t)\,dt,$$

$$a_n = 2\big\langle f, \cos(2\pi nt/T)\big\rangle = \frac{2}{T}\int_0^T f(t)\cos(2\pi nt/T)\,dt \quad \text{for } n \geq 1,$$

$$b_n = 2\langle f, \sin(2\pi nt/T)\rangle = \frac{2}{T}\int_0^T f(t)\sin(2\pi nt/T)\,dt \quad \text{for } n \geq 1.$$

Fourier series of the square wave

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t = linspace(0, T, 100);
y = zeros(size(t));
for n = 1:2:19
    y = y + (4/(n*pi))*sin(2*pi*n*t/T);
end
plot(t,y)
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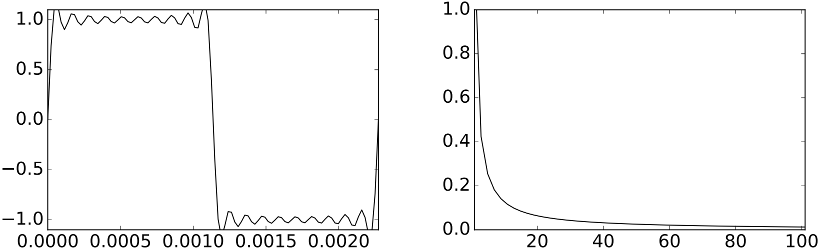


Figure: The Fourier series $(f_s)_{20}$ and the values for the first 100 Fourier coefficients b_n.

Fourier series of the triangle wave

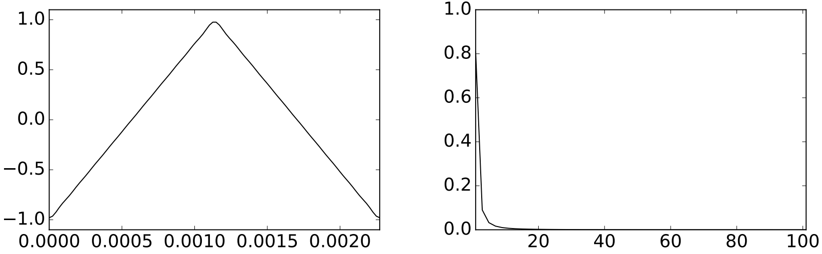


Figure: The Fourier series $(f_t)_{20}$ and the values for the first 100 Fourier coefficients a_n.

Observations

- With $N = 1$, the Fourier series of both the square and the triangle wave are pure tones with frequency $\nu = 1/T$. Largest term.
- f_s is anti-symmetric about 0, and its Fourier series was a sine series.
- f_t is symmetric about 0, and its Fourier series was a cosine series.
- f_t is continuous, and f_s is not.
- The Fourier series of f_t converged faster than that of f_s.

Symmetry and antisymmetry, Theorem 1.10

- If f is antisymmetric about 0 ($f(-t) = -f(t)$ for all t), then $a_n = 0$, so the Fourier series is actually a sine-series.
- If f is symmetric about 0 ($f(-t) = f(t)$ for all t), then $b_n = 0$, so the Fourier series is actually a cosine-series.

Complex Fourier basis, Definition 1.11

We define the set of functions

$$\mathcal{F}_{N,T} = \{e^{-2\pi iNt/T}, e^{-2\pi i(N-1)t/T}, \cdots, e^{-2\pi it/T}, \\ 1, e^{2\pi it/T}, \cdots, e^{2\pi i(N-1)t/T}, e^{2\pi iNt/T}\},$$

and call this the order N *complex Fourier basis* for $V_{N,T}$.

Complex vector spaces and inner products

For general complex functions we extend the definition of the inner product as

$$\langle f, g \rangle = \frac{1}{T} \int_0^T f \, \bar{g} \, dt.$$

The associated norm is

$$\|f\| = \sqrt{\frac{1}{T} \int_0^T |f(t)|^2 dt}. \quad (1)$$

Complex Fourier coefficients, Theorem 1.12

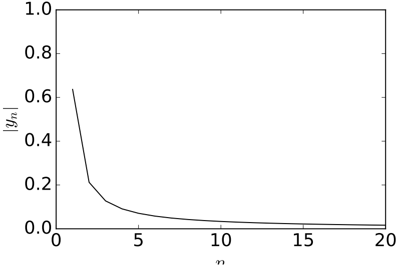
We denote by $y_{-N}, \ldots, y_0, \ldots, y_N$ the coordinates of f_N in the basis $\mathcal{F}_{N,T}$, i.e.

$$f_N(t) = \sum_{n=-N}^{N} y_n e^{2\pi i n t/T}.$$

The y_n are called the complex Fourier coefficients of f, and they are given by.

$$y_n = \langle f, e^{2\pi i n t/T} \rangle = \frac{1}{T} \int_0^T f(t) e^{-2\pi i n t/T} dt.$$

Distribution of Fourier coefficients in Example 1.24



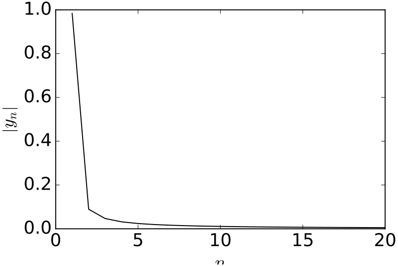


Figure: Plot of $|y_n|$ when $f(t) = e^{2\pi it/T_2}$, and $T_2 > T$. Left: $T/T_2 = 0.5$. Right: $T/T_2 = 0.9$.

Fourier series pairs, Theorem 1.16

we use the notation $f \to y_n$ to indicate that y_n is the n'th (complex) Fourier coefficient of $f(t)$. The functions 1, $e^{2\pi int/T}$, and $\chi_{-a,a}$ have the Fourier coefficients

$$\begin{aligned} 1 &\to \boldsymbol{e}_0 = (1, 0, 0, 0 \ldots,) \\ e^{2\pi int/T} &\to \boldsymbol{e}_n = (0, 0, \ldots, 1, 0, 0, \ldots) \\ \chi_{-a,a} &\to \frac{\sin(2\pi na/T)}{\pi n}. \end{aligned}$$

Fourier series properties, Theorem 1.17

The mapping $f \to y_n$ is linear: if $f \to x_n$, $g \to y_n$, then

$$af + bg \to ax_n + by_n$$

For all n. Moreover, if f is real and periodic with period T, the following properties hold:

1. $y_n = \overline{y_{-n}}$ for all n.
2. If $f(t) = f(-t)$ (i.e. f is symmetric), then all y_n are real, so that b_n are zero and the Fourier series is a cosine series.
3. If $f(t) = -f(-t)$ (i.e. f is antisymmetric), then all y_n are purely imaginary, so that the a_n are zero and the Fourier series is a sine series.
4. If $g(t) = f(t - d)$ (i.e. g is the function f delayed by d) and $f \to y_n$, then $g \to e^{-2\pi ind/T} y_n$.
5. If $g(t) = e^{2\pi idt/T} f(t)$ with d an integer, and $f \to y_n$, then $g \to y_{n-d}$.
6. Let d be a number. If $f \to y_n$, then $f(d + t) = f(d - t)$ for all t if and only if the argument of y_n is $-2\pi nd/T$ for all n.

Convergence of Fourier series 1

Corollary 1.19 If the complex Fourier coefficients of f are y_n and f is differentiable, then the Fourier coefficients of $f'(t)$ are $\frac{2\pi in}{T}y_n$.

Turning this around: the Fourier coefficients of $f(t)$ are $T/(2\pi in)$ times those of $f'(t)$, when f is differentiable. In other words, the Fourier coefficients of a function which is many times differentiable decay to zero very fast.

Observation 1.20 The Fourier series converges quickly when the function is many times differentiable.

Idea 1.21 Assume that f is continuous on $[0, T)$. Can we construct another periodic function which agrees with f on $[0, T]$, and which is both continuous and periodic (maybe with period different from T)?

If this is possible the Fourier series of the new function could produce better approximations for f. It turns out that the following extension strategy does the job:

Definition 1.22: Let f be a function defined on $[0, T]$. By the *symmetric extension* of f, denoted $\breve{f}$, we mean the function defined on $[0, 2T]$ by

$$\breve{f}(t) = \begin{cases} f(t), & \text{if } 0 \leq t \leq T; \\ f(2T - t), & \text{if } T < t \leq 2T. \end{cases}$$

The following holds:

If f is continuous on $[0, T]$, then $\breve{f}$ is continuous on $[0, 2T]$, and $\breve{f}(0) = \breve{f}(2T)$. If we extend $\breve{f}$ to a periodic function on the whole real line (which we also will denote by $\breve{f}$), this function is continuous, agrees with f on $[0, T)$, and is a symmetric function.

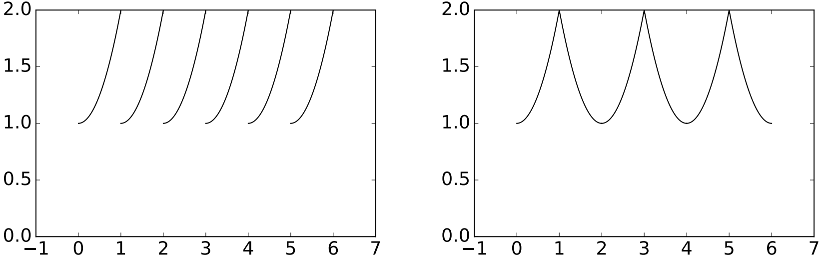


Figure: Two different extensions of f to a periodic function on the whole real line. Periodic extension (left) and symmetric extension (right).

Filters

An operation on sound is called a *filter* if it preserves the different frequencies in the sound. In other words, s is a filter if, for any sound on the form $f = \sum_\nu c(\nu)e^{2\pi i\nu t}$, the output $s(f)$ is a sound which can be written on the form

$$s(f) = s\left(\sum_\nu c(\nu)e^{2\pi i\nu t}\right) = \sum_\nu c(\nu)\lambda_s(\nu)e^{2\pi i\nu t}.$$

$\lambda_s(\nu)$ is a function describing how s treats the different frequencies, and is also called the *frequency response* of s.

Convolution kernels (Theorem 1.25)

Assume that g is a bounded Riemann-integrable function with compact support (i.e. that there exists an interval $[a, b]$ so that $g = 0$ outside $[a, b]$). The operation

$$f(t) \to h(t) = \int_{-\infty}^{\infty} g(u) f(t-u) du. \tag{2}$$

is a filter. Also, the frequency response of the filter is $\lambda_s(\nu) = \int_{\infty}^{\infty} g(s) e^{-2\pi i \nu s} ds$. The function g is also called the *kernel* of s.

Proof of convergence of Fourier series 1

We define two convolution kernels, called the Fejer- and Dirichlet kernels.

$$D_N(t) = \frac{\sin(\pi(2N+1)t/T)}{\sin(\pi t/T)}$$

$$F_N(t) = \frac{1}{N+1}\left(\frac{\sin(\pi(N+1)t/T)}{\sin(\pi t/T)}\right)^2$$

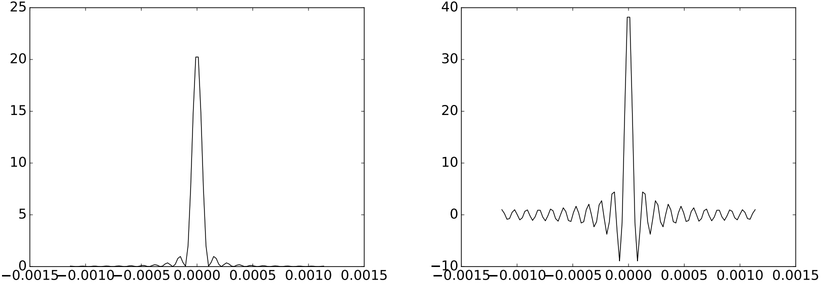


Figure: The Fejer and Dirichlet kernels for $N = 20$.

Proof of convergence of Fourier series 2

Their frequency responses are as follows

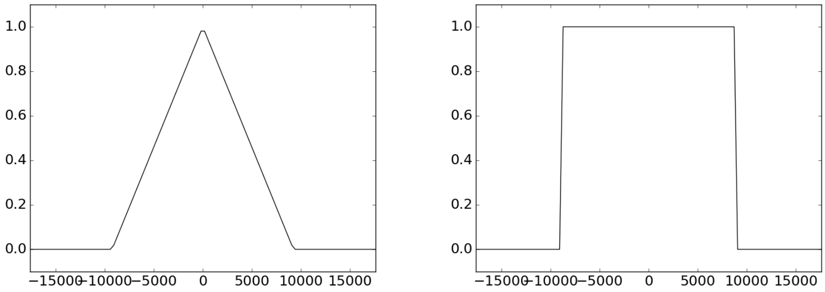
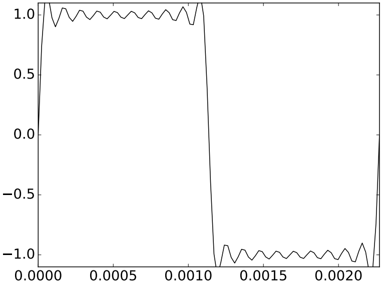


Figure: The frequency responses for the filters with Fejer and Dirichlet kernels, $N = 20$.

Proof of convergence of Fourier series 3

It turns out that filtering with kernel $F_N(t)$ produces $f_N(t) \in V_{N,T}$, while filtering with kernel $D_N(t)$ produces $S_N(t) = \frac{1}{N+1}\sum_{n=0}^{N} f_n(t) \in V_{N,T}$. It also turns out that S_N has much nicer convergence to f than f_N does, and that this convergence is easier to prove.



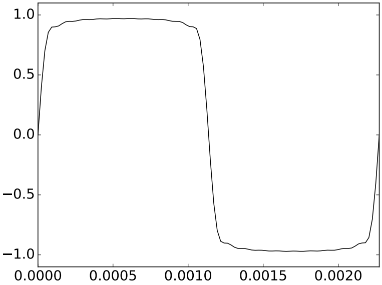


Figure: $f_N(t)$ and $S_N(t)$ for $N = 20$ for the square wave.