

Using tensor products to apply wavelets to images

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Let U_1 and U_2 be vector spaces of functions, defined on the intervals $[0, M)$ and $[0, N)$, respectively, and suppose that $f_1 \in U_1$ and $f_2 \in U_2$. The tensor product of f_1 and f_2 , denoted $f_1 \otimes f_2$, is the function in two variables defined on $[0, M) \times [0, N)$ by

$$(f_1 \otimes f_2)(t_1, t_2) = f_1(t_1)f_2(t_2).$$

The tensor product of the spaces $U_1 \otimes U_2$ is the vector space spanned by the two-variable functions $\{f_1 \otimes f_2\}_{f_1 \in U_1, f_2 \in U_2}$.

$$\langle f, g \rangle = \int_0^N \int_0^M f(t_1, t_2)g(t_1, t_2)dt_1 dt_2.$$

This says that

$$\begin{aligned}\langle f_1 \otimes f_2, g_1 \otimes g_2 \rangle &= \int_0^N \int_0^M f_1(t_1)f_2(t_2)g_1(t_1)g_2(t_2)dt_1 dt_2 \\ &= \int_0^M f_1(t_1)g_1(t_1)dt_1 \int_0^N f_2(t_2)g_2(t_2)dt_2 \\ &= \langle f_1, g_1 \rangle \langle f_2, g_2 \rangle.\end{aligned}$$

This means that for tensor products, a double integral can be computed as the product of two one-dimensional integrals.

Using tensor products for approximation

If U_1 and U_2 can be used to approximate functions in one variable, then $U_1 \otimes U_2$ can be used to approximate functions in two variables.

Tensor products of polynomials: Let $U_1 = U_2$ be the space of all polynomials of finite degree. We know that U_1 can be used for approximating many kinds of functions, such as continuous functions, for example by Taylor series. The tensor product $U_1 \otimes U_1$ consists of all functions on the form $\sum_{i,j} \alpha_{i,j} t_1^i t_2^j$.

Tensor products of Fourier spaces: Let $U_1 = U_2 = V_{N,T}$ be the N th order Fourier space which is spanned by the functions

$$e^{-2\pi i N t/T}, \dots, e^{-2\pi i t/T}, 1, e^{2\pi i t/T}, \dots, e^{2\pi i N t/T}$$

$U_1 \otimes U_1$ now consists of all functions on the form

$$\sum_{k,l=-N}^N \alpha_{k,l} e^{2\pi i k t_1/T} e^{2\pi i l t_2/T}.$$

If $\{f_i\}_{i=0}^{M-1}$ is a basis for U_1 and $\{g_j\}_{j=0}^{N-1}$ is a basis for U_2 , then $\{f_i \otimes g_j\}_{(i,j)=(0,0)}^{(M-1,N-1)}$ is a basis for $U_1 \otimes U_2$. Moreover, if the bases for U_1 and U_2 are orthogonal/orthonormal, then the basis for $U_1 \otimes U_2$ is orthogonal/orthonormal.

if $\mathcal{B} = \{f_i\}_{i=0}^{M-1}$ and $\mathcal{C} = \{g_j\}_{j=0}^{N-1}$, we define $\mathcal{B} \otimes \mathcal{C}$ as the basis $\{f_i \otimes g_j\}_{(i,j)=(0,0)}^{(M-1,N-1)}$ for $U_1 \otimes U_2$. We say that X is the coordinate matrix of f if $f(t_1, t_2) = \sum_{i,j} X_{i,j} (f_i \otimes g_j)(t_1, t_2)$, where $X_{i,j}$ are the elements of X .

Change of coordinates in tensor products of function spaces, Theorem 10.5

Assume that U_1 and U_2 are function spaces, and that

- $\mathcal{B}_1, \mathcal{C}_1$ are bases for U_1 , and that S_1 is the change of coordinates matrix from \mathcal{B}_1 to \mathcal{C}_1 ,
- $\mathcal{B}_2, \mathcal{C}_2$ are bases for U_2 , and that S_2 is the change of coordinates matrix from \mathcal{B}_2 to \mathcal{C}_2 .

Both $\mathcal{B}_1 \otimes \mathcal{B}_2$ and $\mathcal{C}_1 \otimes \mathcal{C}_2$ are bases for $U_1 \otimes U_2$, and if X is the coordinate matrix in $\mathcal{B}_1 \otimes \mathcal{B}_2$, Y the coordinate matrix in $\mathcal{C}_1 \otimes \mathcal{C}_2$, then the change of coordinates from $\mathcal{B}_1 \otimes \mathcal{B}_2$ to $\mathcal{C}_1 \otimes \mathcal{C}_2$ can be computed as

$$Y = S_1 X (S_2)^T.$$

We will now specialize the spaces U_1 , U_2 to resolution spaces V_m and detail spaces W_m . We can in particular form the tensor products $\phi_{0,n_1} \otimes \phi_{0,n_2}$. We assume that

- the first component ϕ_{0,n_1} has period M (so that $\{\phi_{0,n_1}\}_{n_1=0}^{M-1}$ is a basis for the first component space),
- the second component ϕ_{0,n_2} has period N (so that $\{\phi_{0,n_2}\}_{n_2=0}^{N-1}$ is a basis for the second component space).

$V_0 \otimes V_0$ is the MN -dimensional space with basis $\{\phi_{0,n_1} \otimes \phi_{0,n_2}\}_{(n_1,n_2)=(0,0)}^{(M-1,N-1)}$, and the coordinate matrices are $M \times N$. This difference in the dimension of the two components is done to allow for images where the number of rows and columns may be different.

Bases for tensor products

Let ϕ, ψ be a scaling function and a mother wavelet. Then the two sets of tensor products given by

$$\phi_m \otimes \phi_m = \{\phi_{m,n_1} \otimes \phi_{m,n_2}\}_{n_1, n_2}$$

and

$$\begin{aligned} & (\phi_{m-1}, \psi_{m-1}) \otimes (\phi_{m-1}, \psi_{m-1}) \\ &= \{\phi_{m-1,n_1} \otimes \phi_{m-1,n_2}, \\ & \quad \phi_{m-1,n_1} \otimes \psi_{m-1,n_2}, \\ & \quad \psi_{m-1,n_1} \otimes \phi_{m-1,n_2}, \\ & \quad \psi_{m-1,n_1} \otimes \psi_{m-1,n_2}\}_{n_1, n_2} \end{aligned}$$

are both bases for $V_m \otimes V_m$. This second basis is orthogonal/orthonormal whenever the first basis is.

We define the following tensor product spaces:

- The space $W_m^{(0,1)}$ spanned by $\{\phi_{m,n_1} \otimes \psi_{m,n_2}\}_{n_1,n_2}$,
- The space $W_m^{(1,0)}$ spanned by $\{\psi_{m,n_1} \otimes \phi_{m,n_2}\}_{n_1,n_2}$,
- The space $W_m^{(1,1)}$ spanned by $\{\psi_{m,n_1} \otimes \psi_{m,n_2}\}_{n_1,n_2}$.

The coordinate matrix of

$$\begin{aligned}
 & \sum_{n_1, n_2=0}^{2^{m-1}N} (c_{m-1, n_1, n_2} (\phi_{m-1, n_1} \otimes \phi_{m-1, n_2}) + \\
 & \quad w_{m-1, n_1, n_2}^{(0,1)} (\phi_{m-1, n_1} \otimes \psi_{m-1, n_2}) + \\
 & \quad w_{m-1, n_1, n_2}^{(1,0)} (\psi_{m-1, n_1} \otimes \phi_{m-1, n_2}) + \\
 & \quad w_{m-1, n_1, n_2}^{(1,1)} (\psi_{m-1, n_1} \otimes \psi_{m-1, n_2}))
 \end{aligned}$$

in the basis $(\phi_{m-1}, \psi_{m-1}) \otimes (\phi_{m-1}, \psi_{m-1})$ is

$$\left(\begin{array}{cc|cc}
 c_{m-1,0,0} & \cdots & w_{m-1,0,0}^{(0,1)} & \cdots \\
 \vdots & \vdots & \vdots & \vdots \\
 \hline
 w_{m-1,0,0}^{(1,0)} & \cdots & w_{m-1,0,0}^{(1,1)} & \cdots \\
 \vdots & \vdots & \vdots & \vdots
 \end{array} \right) .$$

The coordinate matrix is thus split into four submatrices:

- The c_{m-1} -values, i.e. the coordinates for $V_{m-1} \oplus V_{m-1}$ (the upper left corner).
- The $w_{m-1}^{(0,1)}$ -values, i.e. the coordinates for $W_{m-1}^{(0,1)}$ (the upper right corner).
- The $w_{m-1}^{(1,0)}$ -values, i.e. the coordinates for $W_{m-1}^{(1,0)}$ (the lower left corner).
- The $w_{m-1}^{(1,1)}$ -values, i.e. the coordinates for $W_{m-1}^{(1,1)}$ (the lower right corner).

Implementing the tensor product

Let

$$A_m = P_{(\phi_{m-1}, \psi_{m-1}) \leftarrow \phi_m} \quad B_m = P_{\phi_m \leftarrow (\phi_{m-1}, \psi_{m-1})}$$

be the stages in the DWT and the IDWT, and let

$$X = (c_{m,i,j})_{i,j} \quad Y = \begin{pmatrix} (c_{m-1,i,j})_{i,j} & (w_{m-1,i,j}^{(0,1)})_{i,j} \\ (w_{m-1,i,j}^{(1,0)})_{i,j} & (w_{m-1,i,j}^{(1,1)})_{i,j} \end{pmatrix}$$

be the coordinate matrices in $\phi_m \otimes \phi_m$, and $(\phi_{m-1}, \psi_{m-1}) \otimes (\phi_{m-1}, \psi_{m-1})$, respectively. Then

$$Y = A_m X A_m^T \quad X = B_m Y B_m^T$$

By the m -level two-dimensional DWT/IDWT (or DWT2/IDWT2) we mean the change of coordinates where this is repeated m times as in a DWT/IDWT.

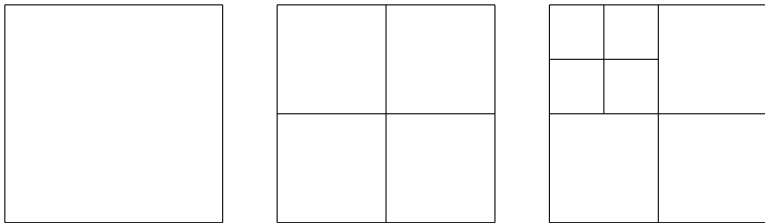


Figure: Illustration of the different coordinates in a two level DWT2 before the first stage is performed (left), after the first stage (middle), and after the second stage (right).

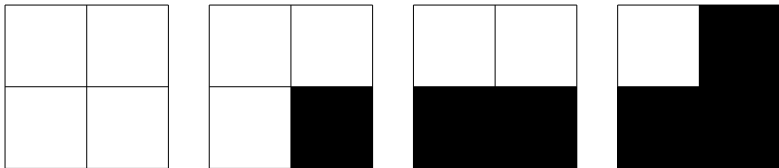


Figure: Graphical representation of neglecting the wavelet coefficients at the first level. After applying DWT2, the wavelet coefficients are split into four parts, as shown in the left figure. In the following figures we have removed coefficients from $W_1^{(1,1)}$, $W_1^{(1,0)}$, and $W_1^{(0,1)}$, in that order.

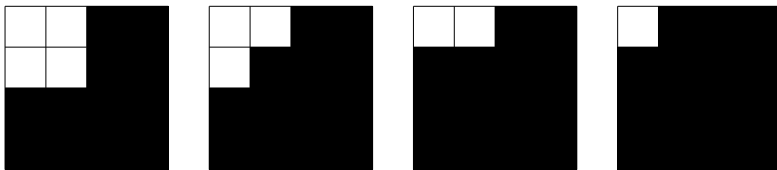


Figure: Graphical representation of neglecting the wavelet coefficients at the second level. After applying the second stage in DWT2, the wavelet coefficients from the upper left corner are also split into four parts, as shown in the left figure. In the following figures we have removed coefficients from $W_2^{(1,1)}$, $W_2^{(1,0)}$, and $W_2^{(0,1)}$, in that order.

Visual interpretation of the DWT2

After the DWT2 has been applied to an image, we expect to see the following:

- In the upper left corner, slow variations in both the vertical and horizontal directions are captured, i.e. this is a low-resolution version of the image.
- In the upper right corner, slow variations in the vertical direction are captured, together with abrupt changes in the horizontal direction.
- In the lower left corner, slow variations in the horizontal direction are captured, together with abrupt changes in the vertical direction.
- In the lower right corner, abrupt changes in both directions appear are captured.

Applying the Haar wavelet to a very simple example image

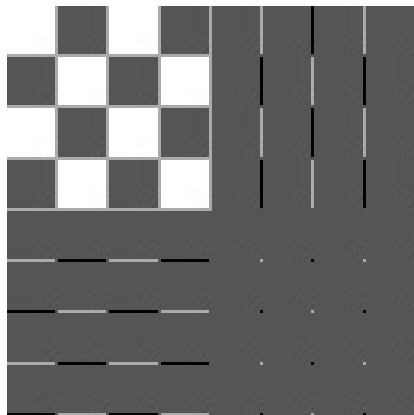


Figure: The chess pattern example image after application of the DWT2.

Creating thumbnail images

We can now use the following code to store a low-resolution approximation for $m = 1$ of half the size:

```
X = DWT2Impl(X, 1, @DWTKernelHaar);  
X = X(1:(size(X,1)/2), 1:(size(X,2)/2), :);  
X = mapto01(X);  
X = X*255;
```

Here it is necessary to map the result back to $[0, 255]$.

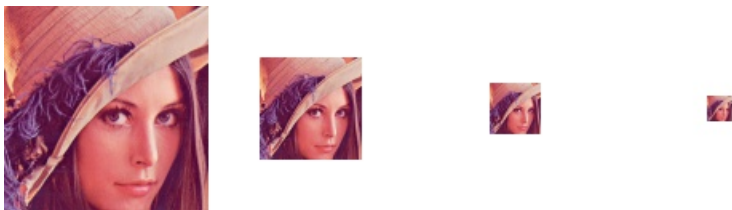


Figure: The corresponding thumbnail images for the Image of Lena, obtained with $m = 1$, $m = 2$, $m = 3$, and $m = 4$.

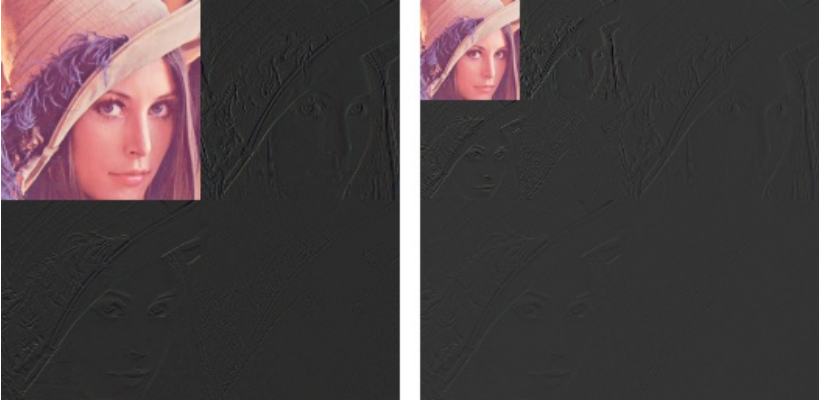


Figure: The corresponding image resulting from a wavelet transform with the Haar-wavelet for $m = 1$ and $m = 2$.

Detail and low-resolution approximations with the Haar wavelet

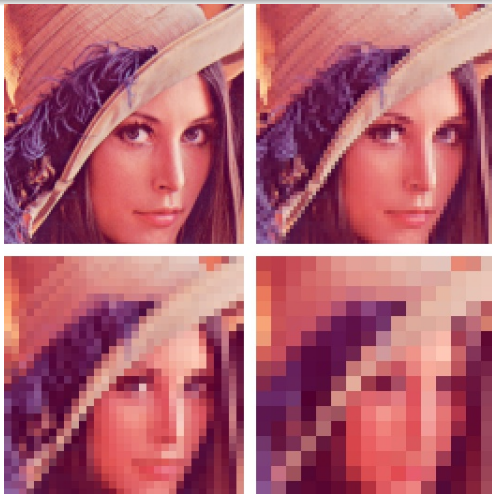


Figure: Image of Lena, with detail at the first 1, 2, 3, and 4 levels zeroed out, respectively.



Figure: The corresponding detail for the images in the previous foil.

The black color indicates values which are close to 0. In other words, most of the coefficients are close to 0.

Displaying the low-resolution approximations at a given resolution

```
showDWT(m, @DWTKernelHaar, @IDWTKernelHaar, 1);  
showDWT(m, @DWTKernel53, @IDWTKernel53, 1);  
showDWT(m, @DWTKernel97, @IDWTKernel97, 1);
```


The Spline 5/3 wavelet and removing bands in the detail spaces

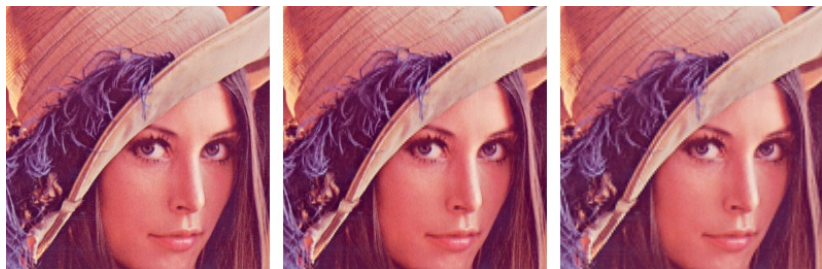


Figure: Image of Lena, with various bands of detail at the first level zeroed out. From left to right, the detail at $W_1^{(1,1)}$, $W_1^{(1,0)}$, $W_1^{(0,1)}$.



Figure: Image of Lena, with various bands of detail at the second level zeroed out. From left to right, the detail at $W_2^{(1,1)}$, $W_2^{(1,0)}$, $W_2^{(0,1)}$.

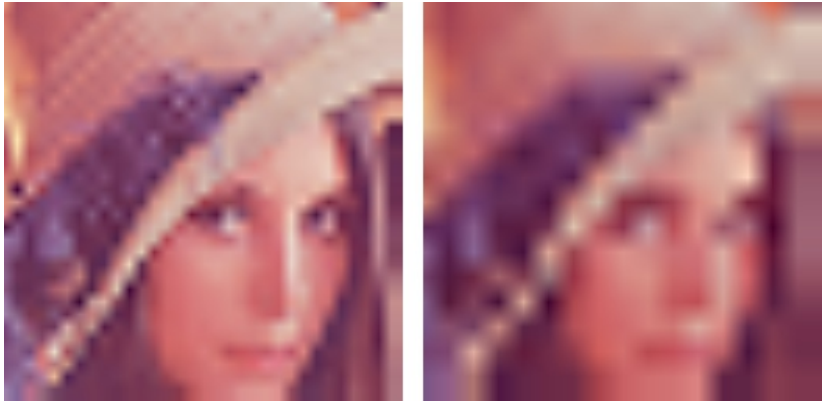


Figure: Image of Lena, with detail including level 3 and 4 zeroed out.



Figure: The corresponding detail for the image of Lena.

The CDF 9/7 wavelet

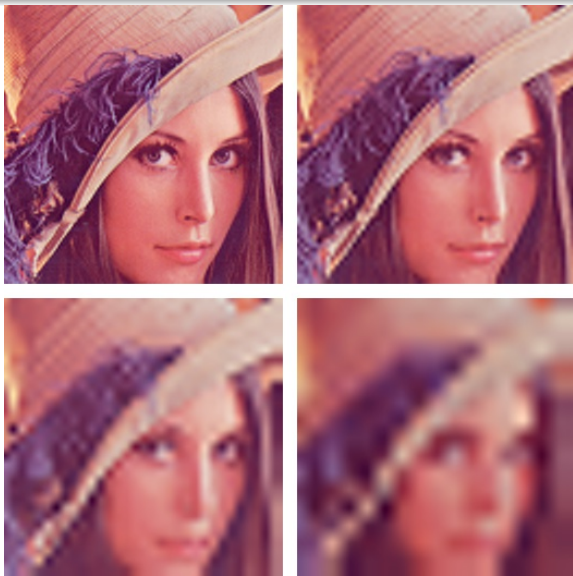


Figure: Image of Lena, with higher levels of detail neglected.

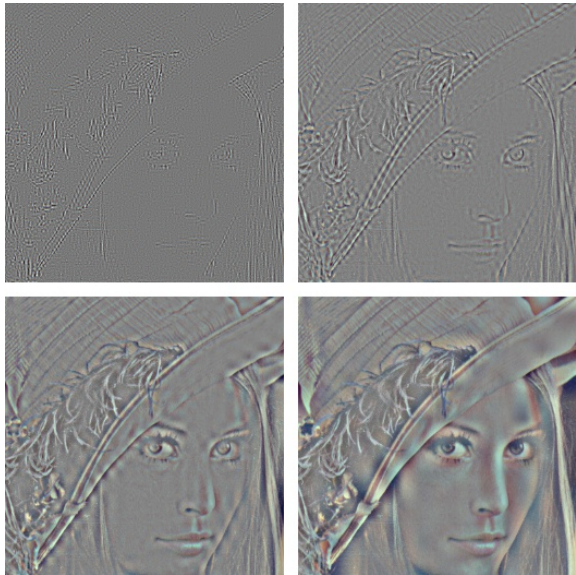


Figure: The corresponding detail for the image of Lena.

An application to the FBI standard for compression of fingerprint images



Figure: A typical fingerprint image.

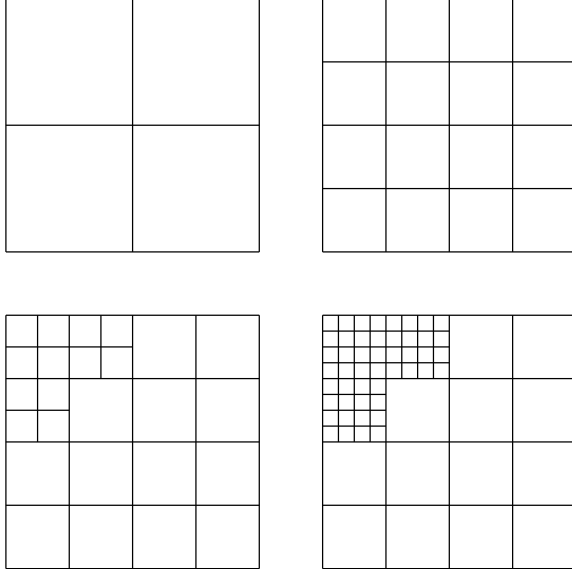


Figure: Subband structure after the different stages of the wavelet applications in the FBI fingerprint compression scheme.

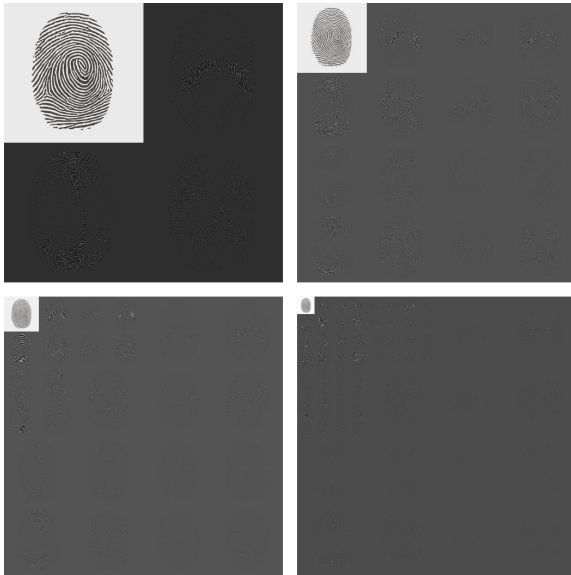


Figure: The fingerprint image after several DWT's.

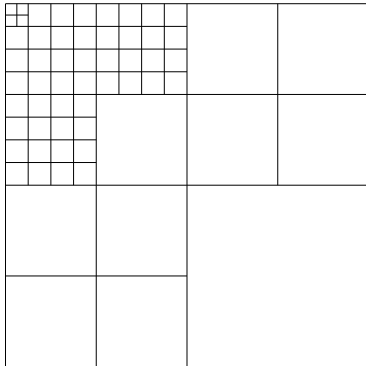


Figure: Subbands structure after all stages.



Figure: The resulting image obtained with the subband decomposition employed by the FBI.

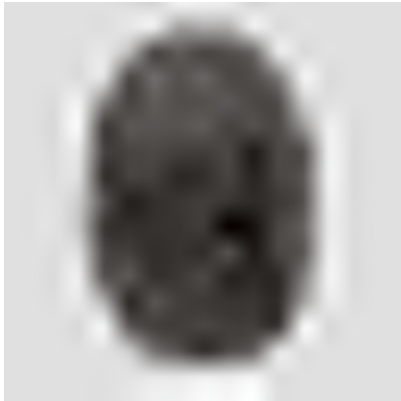


Figure: The low-resolution approximation and the detail obtained by the FBI standard for compression of fingerprint images, when applied to our sample fingerprint image.