Digital sound and discrete Fourier analysis

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Euclidean inner product, Definition 2.1

For complex vectors of length N the Euclidean inner product is given by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{k=0}^{N-1} x_k \overline{y_k}.$$

The associated norm is

$$\|\boldsymbol{x}\| = \sqrt{\sum_{k=0}^{N-1} |x_k|^2}.$$

Discrete Fourier analysis, Definition 2.2

In Discrete Fourier analysis, a vector $\boldsymbol{x} = (x_0, \ldots, x_{N-1})$ is represented as a linear combination of the N vectors

$$\phi_n = \frac{1}{\sqrt{N}} \left(1, e^{2\pi in/N}, e^{2\pi i2n/N}, \ldots, e^{2\pi ikn/N}, \ldots, e^{2\pi in(N-1)/N}\right).$$

These vectors are called the normalised complex exponentials, or the pure digital tones of order N. n is also called frequency index. The whole collection $\mathcal{F}_N = \{\phi_n\}_{n=0}^{N-1}$ is called the N-point Fourier basis.

the N-point Fourier basis is an orthonormal basis for $\mathbb{R}^N$.

Discrete Fourier Transform, Definition 2.4

We will denote the change of coordinates matrix from the standard basis of $\mathbb{R}^N$ to the Fourier basis $\mathcal{F}_N$ by F_N. We will also call this the (N-point) *Fourier matrix*.

The matrix $\sqrt{N}F_N$ is also called the (N-point) *discrete Fourier transform*, or DFT. If $\boldsymbol{x}$ is a vector in R^N, then $\boldsymbol{y} = \mathrm{DFT}\boldsymbol{x}$ are called the DFT coefficients of $\boldsymbol{x}$. (the DFT coefficients are thus the coordinates in $\mathcal{F}_N$, scaled with $\sqrt{N}$). DFT$\boldsymbol{x}$ is sometimes written as $\hat{\boldsymbol{x}}$.

The Fourier matrix is unitary

Theorem 2.5: The Fourier matrix F_N is the unitary $N \times N$-matrix with entries given by

$$(F_N)_{nk} = \frac{1}{\sqrt{N}} e^{-2\pi i n k / N},$$

for $0 \leq n, k \leq N - 1$.

Definition 2.6: The matrix $\overline{F_N}/\sqrt{N}$ is the inverse of the matrix $\mathrm{DFT} = \sqrt{N} F_N$. We call this inverse matrix the *inverse discrete Fourier transform*, or IDFT.

Direct implementation of the DFT in Matlab

```
function y = DFTImpl(x)
    N = size(x, 1);
    y = zeros(size(x));
    for n = 1:N
        D = exp(2*pi*1i*(n-1)*(0:(N-1))/N);
        y(n) = dot(D, x);
    end
```

n has been replaced by $n-1$ in this code since n runs from 1 to N (array indices must start at 1 in Matlab).

Direct implementation of the DFT in Python

```
def DFTImpl(x):
    y = zeros_like(x).astype(complex)
    N = len(x)
    for n in xrange(N):
        D = exp(-2*pi*n*1j*arange(float(N))/N)
        y[n] = dot(D, x)
    return y
```

Properties of the DFT, Theorem 2.7

Let $\boldsymbol{x}$ be a real vector of length N. The DFT has the following properties:

1. $(\widehat{\boldsymbol{x}})_{N-n} = \overline{(\widehat{\boldsymbol{x}})_n}$ for $0 \leq n \leq N-1$.
2. If $x_k = x_{N-k}$ for all n (so $\boldsymbol{x}$ is symmetric), then $\widehat{\boldsymbol{x}}$ is a real vector.
3. If $x_k = -x_{N-k}$ for all k (so $\boldsymbol{x}$ is antisymmetric), then $\widehat{\boldsymbol{x}}$ is a purely imaginary vector.
4. If d is an integer and $\boldsymbol{z}$ is the vector with components $z_k = x_{k-d}$ (the vector $\boldsymbol{x}$ with its elements delayed by d), then $(\widehat{\boldsymbol{z}})_n = e^{-2\pi i dn/N} (\widehat{\boldsymbol{x}})_n$.
5. If d is an integer and $\boldsymbol{z}$ is the vector with components $z_k = e^{2\pi i dk/N} x_k$, then $(\widehat{\boldsymbol{z}})_n = (\widehat{\boldsymbol{x}})_{n-d}$.

Relation between Fourier coefficients and DFT coefficients, Proposition 2.9

Let $N > 2M$, $f \in V_{M,T}$, and let $\boldsymbol{x} = \{f(kT/N)\}_{k=0}^{N-1}$ be N uniform samples from f over $[0, T]$. The Fourier coefficients z_n of f can be computed from

$$(z_0, z_1, \ldots, z_M, \underbrace{0, \ldots, 0}_{N-(2M+1)}, z_{-M}, z_{-M+1}, \ldots, z_{-1}) = \frac{1}{N}\mathrm{DFT}_N \boldsymbol{x}.$$

In particular, the total contribution in f from frequency n/T, for $0 \leq n \leq M$, is given by y_n and y_{N-n}, where $\boldsymbol{y}$ is the DFT of $\boldsymbol{x}$.

Proposition 2.12: Any $f \in V_{M,T}$ can be reconstructed uniquely from a uniform set of samples $\{f(kT/N)\}_{k=0}^{N-1}$, as long as $f_s > 2|\nu|$, where ν denotes the highest frequency in f.

Sampling theorem and the ideal interpolation formula for periodic functions, Theorem 2.13

Let f be a periodic function with period T, and assume that f has no frequencies higher than νHz. Then f can be reconstructed exactly from its samples $f(-MT_s), \ldots, f(MT_s)$ (where T_s is the sampling period, $N = \frac{T}{T_s}$ is the number of samples per period, and $M = 2N+1$) when the sampling rate $f_s = \frac{1}{T_s}$ is bigger than 2ν. Moreover, the reconstruction can be performed through the formula

$$f(t) = \sum_{k=-M}^{M} f(kT_s) \frac{1}{N} \frac{\sin(\pi(t-kT_s)/T_s)}{\sin(\pi(t-kT_s)/T)}.$$

Sampling theorem and the ideal interpolation formula, general version, Theorem 2.14

Assume that f has no frequencies higher than νHz. Then f can be reconstructed exactly from its samples $\ldots, f(-2T_s), f(-T_s), f(0), f(T_s), f(2T_s), \ldots$ when the sampling rate is bigger than 2ν. Moreover, the reconstruction can be performed through the formula

$$f(t) = \sum_{k=-\infty}^{\infty} f(kT_s) \frac{\sin(\pi(t-kT_s)/T_s)}{\pi(t-kT_s)/T_s}.$$

Using the DFT to adjust frequencies in sound, Example 2.16

```
[x, fs] = forw_comp_rev_DFT('L', 13000, 'lower', 1);
playerobj=audioplayer(x, fs);
playblocking(playerobj);
```

Compression by zeroing out DFT coefficients, Example 2.17

```
[x, fs] = forw_comp_rev_DFT('threshold', 20);
playerobj=audioplayer(x, fs);
playblocking(playerobj);
```

```
[x, fs] = forw_comp_rev_DFT('n', 3);
playerobj=audioplayer(x, fs);
playblocking(playerobj);
```

FFT algorithm when N is even Theorem 2.15

Let $\boldsymbol{y} = \mathrm{DFT}_N \boldsymbol{x}$ be the N-point DFT of $\boldsymbol{x}$, with N an even number, and let $D_{N/2}$ be the $(N/2) \times (N/2)$-diagonal matrix with entries $(D_{N/2})_{n,n} = e^{-2\pi i n/N}$ for $0 \leq n < N/2$. Then we have that

$$
\begin{aligned}
(y_0, y_1, \ldots, y_{N/2-1}) &= \mathrm{DFT}_{N/2} \boldsymbol{x}^{(e)} + D_{N/2} \mathrm{DFT}_{N/2} \boldsymbol{x}^{(o)} \\
(y_{N/2}, y_{N/2+1}, \ldots, y_{N-1}) &= \mathrm{DFT}_{N/2} \boldsymbol{x}^{(e)} - D_{N/2} \mathrm{DFT}_{N/2} \boldsymbol{x}^{(o)}
\end{aligned}
$$

where $\boldsymbol{x}^{(e)}, \boldsymbol{x}^{(o)} \in \mathbb{R}^{N/2}$ consist of the even- and odd-indexed entries of $\boldsymbol{x}$, respectively, i.e.

$$
\boldsymbol{x}^{(e)} = (x_0, x_2, \ldots, x_{N-2}) \qquad \boldsymbol{x}^{(o)} = (x_1, x_3, \ldots, x_{N-1}).
$$

IFFT algorithm when N is even, Theorem 2.15

Let N be an even number and let $\tilde{\boldsymbol{x}} = \overline{\mathrm{DFT}_N}\boldsymbol{y}$. Then we have that

$$
\begin{aligned}
(\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_{N/2-1}) &= \overline{\mathrm{DFT}_{N/2}}\boldsymbol{y}^{(e)} + \overline{D_{N/2}\mathrm{DFT}_{N/2}})\boldsymbol{y}^{(o)} \\
(\tilde{x}_{N/2}, \tilde{x}_{N/2+1}, \ldots, \tilde{x}_{N-1}) &= \overline{\mathrm{DFT}_{N/2}}\boldsymbol{y}^{(e)} - \overline{D_{N/2}\mathrm{DFT}_{N/2}})\boldsymbol{y}^{(o)}
\end{aligned}
$$

where $\boldsymbol{y}^{(e)}, \boldsymbol{y}^{(o)} \in \mathbb{R}^{N/2}$ are the vectors

$$
\boldsymbol{y}^{(e)} = (y_0, y_2, \ldots, y_{N-2}) \qquad \boldsymbol{y}^{(o)} = (y_1, y_3, \ldots, y_{N-1}).
$$

Moreover, $\boldsymbol{x} = \mathrm{IDFT}_N\boldsymbol{y}$ can be computed from
$\boldsymbol{x} = \tilde{\boldsymbol{x}}/N = \overline{\mathrm{DFT}_N}\boldsymbol{y}/N$

DFT and IDFT matrix factorizations, theorem 2.18

We have that

$$
\begin{aligned}
\mathrm{DFT}_N \boldsymbol{x} &= \begin{pmatrix} I & D_{N/2} \\ I & -D_{N/2} \end{pmatrix} \begin{pmatrix} \mathrm{DFT}_{N/2} & \mathbf{0} \\ \mathbf{0} & \mathrm{DFT}_{N/2} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}^{(e)} \\ \boldsymbol{x}^{(o)} \end{pmatrix} \\
\mathrm{IDFT}_N \boldsymbol{y} &= \frac{1}{N} \overline{\begin{pmatrix} I & D_{N/2} \\ I & -D_{N/2} \end{pmatrix}} \begin{pmatrix} \overline{\mathrm{DFT}_{N/2}} & \mathbf{0} \\ \mathbf{0} & \overline{\mathrm{DFT}_{N/2}} \end{pmatrix} \begin{pmatrix} \boldsymbol{y}^{(e)} \\ \boldsymbol{y}^{(o)} \end{pmatrix}
\end{aligned}
$$

Iterating the factorization 1

$$
\mathrm{DFT}_N \boldsymbol{x} = \begin{pmatrix} I & D_{N/2} \\ I & -D_{N/2} \end{pmatrix} \begin{pmatrix} I & D_{N/4} & \mathbf{0} & \mathbf{0} \\ I & -D_{N/4} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I & D_{N/4} \\ \mathbf{0} & \mathbf{0} & I & -D_{N/4} \end{pmatrix} \times
$$

$$
\begin{pmatrix} \mathrm{DFT}_{N/4} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathrm{DFT}_{N/4} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathrm{DFT}_{N/4} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathrm{DFT}_{N/4} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}^{(ee)} \\ \boldsymbol{x}^{(eo)} \\ \boldsymbol{x}^{(oe)} \\ \boldsymbol{x}^{(oo)} \end{pmatrix}
$$

where the vectors $\boldsymbol{x}^{(e)}$ and $\boldsymbol{x}^{(o)}$ have been further split into even- and odd-indexed entries. Clearly, if this factorization is repeated, we obtain a factorization

Iterating the factorization 2

$$\mathrm{DFT}_N = \prod_{k=1}^{\log_2 N} \begin{pmatrix} I & D_{N/2^k} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ I & -D_{N/2^k} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I & D_{N/2^k} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I & -D_{N/2^k} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & I & D_{N/2^k} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & I & -D_{N/2^k} \end{pmatrix} P. \tag{1}$$

FFT implementation

```
function y = FFTImpl(x, FFTKernel)
    x = bitreverse(x);
    y = FFTKernel(x);
```

```
function y = FFTKernelStandard(x)
    N = size(x, 1);
    if N == 1
        y = x;
    else
        xe = FFTKernelStandard(x(1:(N/2)));
        xo = FFTKernelStandard(x((N/2+1):N));
        D = exp(-2*pi*1j*(0:(N/2-1))'/N);
        xo = xo.*D;
        y = [ xe + xo; xe - xo];
    end
```

```
y = FFTImpl(x, @FFTKernelStandard);
```