The filter representation of wavelets

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MRA-matrices, Definition 6.1

An $N \times N$-matrix T, with N even, is called an MRA-matrix if the columns are translates of the first two columns in alternating order, in the same way as the columns of a circulant Toeplitz matrix.

H_0 and H_1, Definition 6.2

We denote by H_0 the (unique) filter with the same first row as H, and by H_1 the (unique) filter with the same second row as H. H_0 and H_1 are also called the *DWT filter components*.

DWT expressed in terms of filters, Theorem 6.3

Let $\boldsymbol{c}_m$ be the coordinates in ϕ_m, and let H_0, H_1 be defined as above. Any stage in a DWT can ble implemented in terms of filters as follows:

- Compute $H_0\boldsymbol{c}_m$. The even-indexed entries in the result are the cordinates $\boldsymbol{c}_{m-1}$ in ϕ_{m-1}.
- Compute $H_1\boldsymbol{c}_m$. The odd-indexed entries in the result are the coordinates $\boldsymbol{w}_{m-1}$ in ψ_{m-1}.

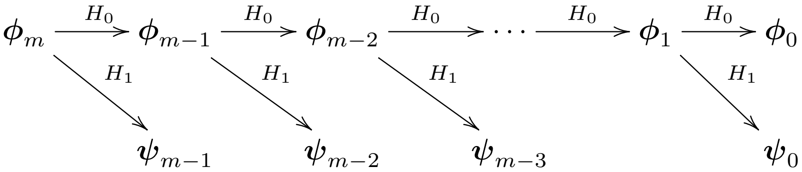


Figure: Detailed illustration of a wavelet transform.

G_0 and G_1, Definition 6.4

We denote by G_0 the (unique) filter with the same first column as G, and by G_1 the (unique) filter with the same second column as G. G_0 and G_1 are also called the *IDWT filter components*.

IDWT expressed in terms of filters, Theorem 6.5

Let G_0, G_1 be defined as above. Any stage in an IDWT can be implemented in terms of filters as follows:

$$\boldsymbol{c}_m = G_0 \begin{pmatrix} c_{m-1,0} \\ 0 \\ c_{m-1,1} \\ 0 \\ \cdots \\ c_{m-1,2^{m-1}N-1} \\ 0 \end{pmatrix} + G_1 \begin{pmatrix} 0 \\ w_{m-1,0} \\ 0 \\ w_{m-1,1} \\ \cdots \\ 0 \\ w_{m-1,2^{m-1}N-1} \end{pmatrix}.$$

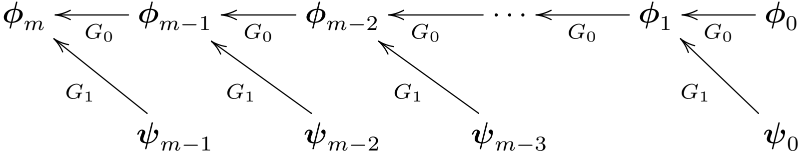


Figure: Detailed illustration of an IDWT.

Computing DWT/IDWT through filters, Fact 6.6

The DWT can be computed with the help of two filters H_0, H_1, as explained in Theorem 6.3. Any linear transformation computed from two filters H_0, H_1 in this way is called a *forward filter bank transform*. The IDWT can be computed with the help of two filters G_0, G_1 as explained in Theorem 6.5. Any linear transformation computed from two filters G_0, G_1 in this way is called a *reverse filter bank transform*.

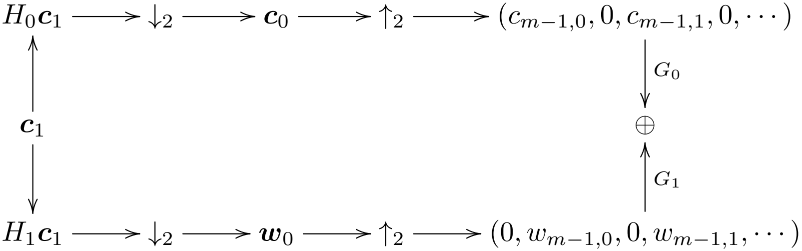


Figure: Detailed illustration of DWT and IDWT.

Dual filter bank transforms, Definition 6.7

Assume that H_0, H_1 are the filters of a forward filter bank transform, and that G_0, G_1 are the filters of a reverse filter bank transform. By the *dual transforms* we mean the forward filter bank transform with filters $(G_0)^T, (G_1)^T$, and the reverse filter bank transform with filters $(H_0)^T, (H_1)^T$.

Fact about the `dual`-parameter in the DWT:

- If the `dual` parameter is false, the DWT is computed as the forward filter bank transform with filters H_0, H_1, and the IDWT is computed as the reverse filter bank transform with filters G_0, G_1.
- If the `dual` parameter is true, the DWT is computed as the forward filter bank transform with filters $(G_0)^T, (G_1)^T$, and the IDWT is computed as the reverse filter bank transform with filters $(H_0)^T, (H_1)^T$.

Frequency responses

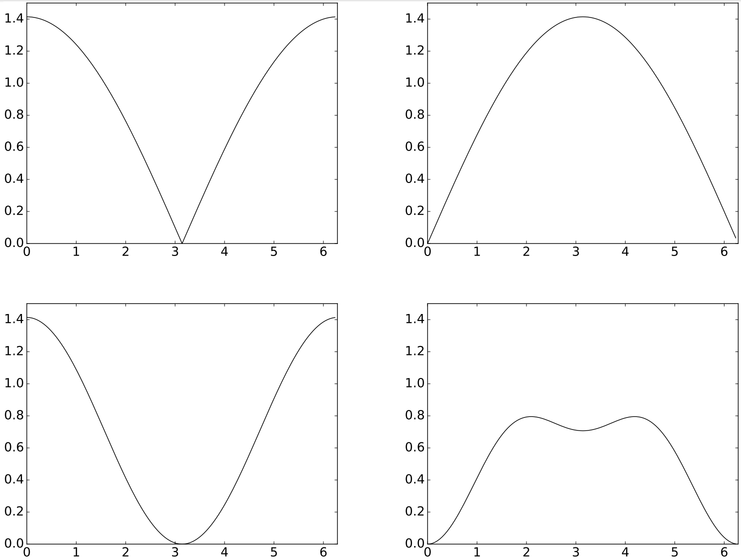
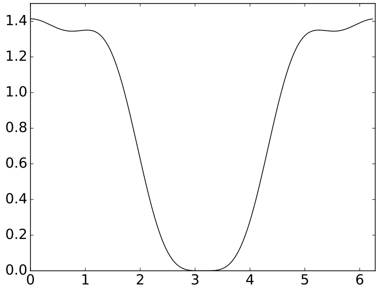


Figure: The frequency responses $\lambda_{G_0}(\omega)$ and $\lambda_{G_1}(\omega)$ for the Haar wavelet (top), and for the alternative piecewise linear wavelet (bottom).

Other frequency responses



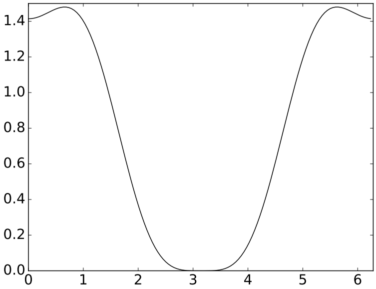


Figure: The frequency responses $\lambda_{H_0}(\omega)$ (left) and $\lambda_{G_0}(\omega)$ (right) for the CDF 9/7 wavelet.

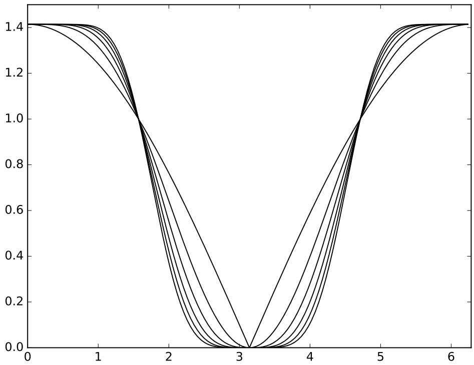


Figure: The magnitudes $|\lambda_{G_0}(\omega)| = |\lambda_{H_0}(\omega)|$ for the first orthonormal wavelets.

Forward filter bank transform, Definition 6.21

Let $H_0, H_1, \ldots, H_{M-1}$ be $N \times N$-filters. A *forward filter bank transform* H produces output $\boldsymbol{z} \in \mathbb{R}^N$ from the input $\boldsymbol{x} \in \mathbb{R}^N$ in the following way:

- $z_{iM} = (H_0\boldsymbol{x})_{iM}$ for any i so that $0 \leq iM < N$.
- $z_{iM+1} = (H_1\boldsymbol{x})_{iM+1}$ for any i so that $0 \leq iM + 1 < N$.
- ...
- $z_{iM+(M-1)} = (H_{M-1}\boldsymbol{x})_{iM+(M-1)}$ for any i so that $0 \leq iM + (M-1) < N$.

In other words, the output of a forward filter bank transform is computed by applying filters $H_0, H_1, \ldots, H_{M-1}$ to the input, and by downsampling and assembling these so that we obtain the same number of output samples as input samples (also in this more general setting this is called critical sampling). $H_0, H_1, \ldots, H_{M-1}$ are also called *analysis filter components*, the output of filter H_i is called channel i *channel*, and M is called the number of channels. The output samples z_{iM+k} are also called the *subband samples* of channel k.

Reverse filter bank transforms, Definition 6.22

Let $G_0, G_1, \ldots, G_{M-1}$ be $N \times N$-filters. An *reverse filter bank transform* G produces $\boldsymbol{x} \in \mathbb{R}^N$ from $\boldsymbol{z} \in \mathbb{R}^N$ in the following way:

- Define $\boldsymbol{z}_k \mathbb{R}^N$ as the vector where $(\boldsymbol{z}_k)_{iM+k} = \boldsymbol{z}_{iM+k}$ for all i so that $0 \leq iM + k < N$, and $(\boldsymbol{z}_k)_s = 0$ for all other s.

$$\boldsymbol{x} = G_0 \boldsymbol{z}_0 + G_1 \boldsymbol{z}_1 + \ldots + G_{M-1} \boldsymbol{z}_{M-1}.$$

$G_0, G_1, \ldots, G_{M-1}$ are also called *synthesis filter components*.

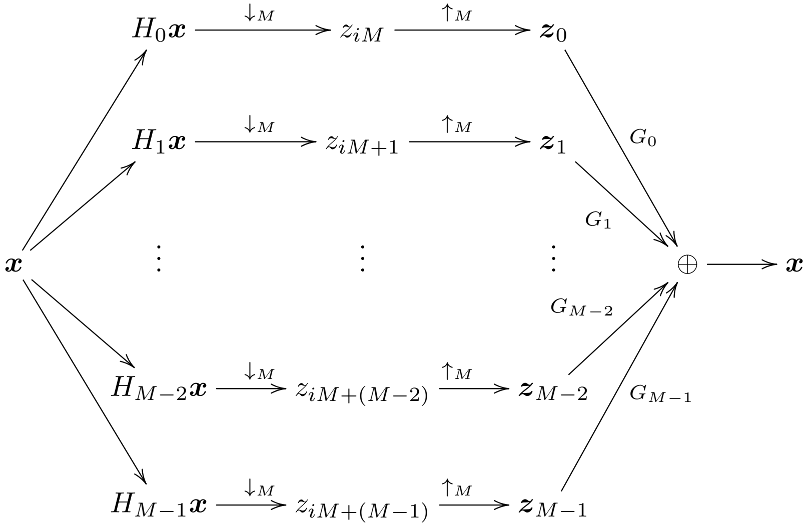


Figure: Illustration of forward and reverse filter bank transforms.