Network flows and combinatorial matrix theory

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Abstract

This note gives a very brief introduction to the theory of network flows and some related topics in combinatorial matrix theory.

1 Network flow theory

Network flow theory is a mathematical treatment of flows in networks. Actually, the classical and basic treatment of this area is the pioneering monograph by L.R. Ford and D.R. Fulkerson called *Flows in networks* [9]. A more recent, and highly recommended, book is [1] which treats both theoretical and computational aspects of this area. It also presents a lot of interesting applications. A comprehensive text in this area and combinatorial optimization more generally, covering basically all theoretical and algorithmic aspects, is the book (three volumes) by Lex Schrijver [18]. Some other interesting books on network flows and optimization are [3] and [16].

1.1 Flows and circulations

Let $D = (V, E)$ be a (directed) *graph* with vertex set V and edge set E. This means that V is a finite set and E is a (finite) set of ordered pairs of distinct elements from V. Each element $v \in V$ is called a *vertex*, and each element $e = (u, v) \in E$ is called an *edge* (or directed edge). Graphs arise in many applications. For instance, they are used to represent transportation or communication networks. Each graph may be represented by a drawing in the plane where vertices and edges correspond to points and lines (curves) between points (using an arrow to indicate the direction: the edge $e = (u, v)$ goes from u to v). Let $n = |V|$ and $m = |E|$ be the number of vertices and edges, respectively.

Certain edge sets are of special interest. For each vertex v define

- $\delta^+(v) = \{e \in E : e = (v, w)$ for some vertex $w \in V\}$: the set of edges leaving v
- $\delta^-(v) = \{e \in E : e = (u, v)$ for some vertex $u \in V\}$: the set of edges entering v

We are interested in functions whose domain are V or E. By enumerating the elements in V and E these functions may be identified with vectors (in $\mathbb{R}^n$ or $\mathbb{R}^m$) containing the functions values. A *flow* is (simply) a function $x : E \to \mathbb{R}$, i.e., $x \in \mathbb{R}^E$ (in general, $\mathbb{R}^S$ denotes the vector space of all real-valued functions with domain is a set S). So x assigns a value $x(e)$ to each edge e; it is called the flow in that edge. Usually we require this flow to be nonnegative, so $x : E \to \mathbb{R}_+$. A flow x gives rise to another function whose domain is V. Let the function $\text{div}_x : V \to \mathbb{R}$ be given by

$$\text{div}_x(v) = \sum_{e \in \delta^+(v)} x(e) - \sum_{e \in \delta^-(v)} x(e).$$

This linear function – which clearly depends on x – is called the *divergence* of x. It gives the difference between total outflow and total inflow in every vertex v. In general

$$\sum_{v \in V} \text{div}_x(v) = 0$$

which we leave as a small exercise to prove.

We are mainly interested in flows with a given divergence. Let $b : V \to \mathbb{R}$ be a given function satisfying $\sum_{v \in V} b(v) = 0$. A flow x with $\text{div}_x = b$ therefore satisfies

$$\sum_{e \in \delta^+(v)} x(e) - \sum_{e \in \delta^-(v)} x(e) = b(v) \qquad (v \in V). \tag{1}$$

In network optimization such flows with given divergence are the central objects. The linear equations (1) are called *flow balance equations*. The set of flows with a given divergence b is therefore a polyhedron[1]. We shall say more about the structure of this polyhedron later, for instance it is possible to describe all its vertices.

[1] Recall that a *polyhedron* is the solution set of a system of (finitely many) linear inequalities in p variables.

Let O denote the the zero function; the constant function with function value 0 (defined on a suitable set). We may also use O to denote the zero vector. A *circulation* in D is a flow x with $\mathrm{div}_x = O$. Thus, in a circulation, the total inflow equals the total outflow in every vertex. This property is usually called *flow conservation*, and it is basic in many applications. For instance, flow conservation holds in junctions in traffic networks or oil pipeline networks. The role of b (in a flow with divergence b) is to specify possible supply or demand in the vertices.

Often we have additional constraints on the flows. The most basic such constraint is to have lower and upper bounds on the flow in each edge. In order to represent these bounds, consider a nonnegative function $c : E \to \mathbb{R}_+$, called the *capacity function*. The constraint on the flow is then

$$O \leq x \leq c$$

which means that

$$0 \leq x(e) \leq c(e) \qquad (e \in E)$$

Sometimes we are interested in a lower bound which is nonzero, positive or negative, but often in applications the lower bound is zero as above.

A (directed) *path* in a graph D is an alternating sequence of distinct vertices and edges

$$P : v_0, e_1, v_1, e_2, \ldots, e_t, v_t$$

where $e_i = (v_{i-1}, v_i)$ $(1 \leq i \leq t)$. Note that we require the vertices (and edges) to be distinct; if repetition of these are allowed we obtain a *walk*. Sometimes a path is just viewed as a vertex sequence or an edge sequence. We also call P a v_0v_t*-path*, and say that P is a path from v_0 to v_t.

Finally, a word on our notation: for any edge function $h : E \to \mathbb{R}$, we may write $h(u, v)$ instead of $h(e)$ when $e = (u, v)$.

1.2 Existence of circulations and flows

A given graph may, or may not, have a flow with specified divergence and, possibly, satisfying capacity constraints. It is therefore natural to look for characterizations of the existence of such flows. One might say that such existence theorems represent the mathematical core of network flow theory. From a more applied point of view these results are closely related to efficient algorithms for finding feasible flows or even finding optimal flows (in network optimization).

We introduce a notion which corresponds to the “boundary” of a vertex subset. Let $S \subseteq V$ and define

- $\delta^+(S) = \{e \in E : e = (v,w), v \in S, w \notin S\}$: the set of edges leaving S
- $\delta^-(S) = \{e \in E : e = (v,w), v \notin S, w \in S\}$: the set of edges entering S

This notation is consistent with our previous notation, e.g., $\delta^+(\{v\}) = \delta^+(v)$.

The next theorem is due to Alan Hoffman (1960) [11] and it characterizes when a graph has a circulation satisfying lower and upper bounds on the flow in each edge. We give a proof following the recommended lecture notes [17] by Lex Schrijver, see also [18]. For a pair $e = (u,v)$ define the "inverse edge" $e^{-1} = (v,u)$. Consider a graph D and functions l, x and u from E into $\mathbb{R}$ such that $l \le x \le u$. Next, define

$$E_x = \{e \in E : x(e) < u(e)\} \cup \{e^{-1} : e \in E,\ l(e) < x(e)\}$$

and the graph $D_x = (V, E_x)$. So, for instance, if $l(e) < x(e) < u(e)$, the auxiliary graph contains both e and e^{-1}. We call D_x the *auxiliary graph* associated with the flow x. This notion is useful in the next proof, and it is also very useful for algorithms in network flows.

Theorem 1.1 (Hoffman's circulation theorem) *Let $l, u : E \to \mathbb{R}$ be edge functions satisfying $l \le u$. Then there exists a circulation x in D such that*

$$l \le x \le u$$

if and only if

$$\sum_{e \in \delta^-(S)} l(e) \le \sum_{e \in \delta^+(S)} u(e) \qquad (S \subseteq V). \tag{2}$$

Moreover, if l and u are integral (the function values are integral), then x can be taken to be integral.

Proof. Assume that x is a circulation with $l \le x \le u$. Then

$$\sum_{e \in \delta^-(S)} l(e) \le \sum_{e \in \delta^-(S)} x(e) = \sum_{e \in \delta^+(S)} x(e) \le \sum_{e \in \delta^+(S)} u(e)$$

where the equality follows from the fact that x is a circulation (see Exercises).

To prove sufficiency, assume that (2) holds. Let $x : E \to \mathbb{R}$ be a function satisfying $l \le x \le u$ such that $\|\mathrm{div}_x\|_1$ is minimized (such an x exists by the Extreme value theorem in analysis). Define

$$V^- = \{v \in V : \mathrm{div}_x(v) < 0\},\ \ V^+ = \{v \in V : \mathrm{div}_x(v) > 0\}.$$

If $V^- = \emptyset$, then ($V^+ = \emptyset$ and) x is a circulation as desired. So assume that V^- is nonempty; we shall deduce a contradiction from this. If the auxiliary graph $D_x = (V, E_x)$ contains a path from (a vertex in) V^- to (a vertex in) V^+, then we can modify x along P (by adding some small ϵ on each edge) and get another flow z with $l \leq z \leq u$ and $\|\mathrm{div}_z\|_1 < \|\mathrm{div}_x\|_1$. Therefore, we may assume that no such path exists. Let S be the set of vertices reachable in D_x from a vertex in V^-. Then for each $e \in \delta^+(S)$ (these edges are in D), we have $e \notin D_x$, and therefore $x(e) = u(e)$. Similarly, for each $e \in \delta^-(S)$, we have $e^{-1} \notin D_x$, so $x(e) = l(e)$. This gives

$$\begin{aligned}
\textstyle\sum_{e\in\delta^+(S)} u(e) - \sum_{e\in\delta^-(S)} l(e) &= \textstyle\sum_{e\in\delta^+(S)} x(e) - \sum_{e\in\delta^-(S)} x(e) \\
&= \textstyle\sum_{v\in S} \mathrm{div}_x(v) \\
&= \textstyle\sum_{v\in V^-} \mathrm{div}_x(v) \\
&< 0.
\end{aligned}$$

But this contradicts (2), so x is a desired circulation. The integrality result may be shown by starting with the zero circulation and keep minimizing the norm of the divergence by modifying the flow along a path as descibed above. This procedure maintains integrality when l and u are integral, and the final circulation has the desired properties. □

Theorem 1.1 may be used to derive other existence results for flows. The technique here is to modify the graph suitably, apply the known result and interpret the result back in the original problem. This is done in the proof of the following basic existence theorem for network flows. The characterization is in terms of an inequality for each vertex subset S of V.

Theorem 1.2 (Existence of flows) *In a directed graph $D = (V, E)$ let $b : V \to \mathbb{R}$ be a supply function and $c : E \to \mathbb{R}_+$ an edge capacity function. Then there is a flow x with divergence b and satisfying $O \leq x \leq c$ if and only if*

$$\begin{aligned}
&\textstyle\sum_{v\in V} b(v) = 0, \quad \text{and} \\
&\textstyle\sum_{v\in S} b(v) \leq \sum_{e\in\delta^+(S)} c(e) \quad (S \subseteq V).
\end{aligned} \tag{3}$$

Proof. Note first that $\sum_v b(v) = 0$ is a necessary condition for a flow to exist; just sum all the flow balance equations (1). Define $V^- = \{v \in V : b(v) < 0\}$ and $V^+ = \{v \in V : b(v) > 0\}$. Construct a graph $D' = (V', E')$ with vertex set $V' = V \cup \{s\}$, so we add a new vertex s. Let the edge set consist of (i) each edge $e \in E$, (ii) an edge (s, v) for each $v \in V^+$, and (iii) an edge (v, s) for each $v \in V^-$. Define $l, u : E' \to \mathbb{R}$ by $l(s, v) = u(s, v) = b(v)$ for each $v \in V^+$, $l(v, s) = u(v, s) = -b(v)$ for each $v \in V^-$, while $l(e) = 0$ and

$u(e) = c(e)$ for each $e \in E$. Consider a circulation $x \in \mathbb{R}^{E'}$ in D' satisfying $l \leq x \leq u$. Then $x(s, v) = b(v)$ for each $v \in V^+$, and $x(v, s) = -b(v)$ for each $v \in V^-$. Then the restriction of x to E satisfies the flow balance constraints (1), and $0 \leq x(e) \leq c(e)$ for all $e \in E$. This defines a bijection between circulations in D' and flows in D, satisfying the respective bounds on the variables. (Note that flow balance of the circulation in vertex s corresponds to $\sum_v b(v) = 0$.) We now apply Theorem 1.1 to D' and the functions l, u. Let $S \subseteq V'$. Then (2) gives

$$\sum_{v \in S \cap V^+} b(v) \leq \sum_{v \in S \cap V^-} (-b(v)) + \sum_{e \in \delta^+(S)} c(e)$$

which is equivalent to (3), and the theorem follows. $\square$

We remark that the construction (of the new graph D') used in the previous proof is also useful computationally for deciding if there is a flow satisfying given divergence and capacity constraints. Actually, if it exists one also wants to find such a flow. This task may be done (essentially in D') by solving a maximum flow problem. And the maximum flow problem is the topic of the next section!

1.3 Maximum flow and minimum cut

We now consider two important optimization problems in digraphs: the maximum flow problem and the minimum cut problem. There are strong connections between these two problems, so it is natural to discuss them in parallel.

Let $D = (V, E)$ be a directed graph with nonnegative edge capacity function $c : E \to \mathbb{R}_+$. Let s and t be two distinct vertices, called the source and the sink, respectively. An *st-flow* is a function $x : E \to \mathbb{R}$ (a flow) satisfying

$$\begin{array}{c} \sum_{e \in \delta^+(v)} x(e) = \sum_{e \in \delta^-(v)} x(e) \quad (v \in V \setminus \{s, t\}) \\ O \leq x \leq c. \end{array} \tag{4}$$

(A more accurate notion would be "an st-flow under c", but we simply use the term st-flow.) So flow conservation holds in all vertices except s and t. The *value* of an st-flow x is defined as

$$\text{val}(x) = \sum_{e \in \delta^+(s)} x(e)$$

which is the total outflow from the source. This defines a linear function val : $\mathbb{R}^E \to \mathbb{R}$. Actually, if D contains no edge entering s (as we may assume without loss of generality in this context) then $\mathrm{val}(x) = \mathrm{div}_x(s)$. The *maximum flow problem* is to find an st-flow x which maximizes $\mathrm{val}(x)$; such a flow is called a *maximum flow*. This is a linear programming[2] problem, which implies that a maximum flow really exists. This existence also follows from the Extreme value theorem: we maximize a continuous function over a compact set. But the maximum flow problem may also be treated – and solved – by combinatorial methods as we shall see below.

We now present the second optimization problem. An *st-cut* K is an edge subset of the form $K = \delta^+(S)$ for a vertex set $S \subseteq V$ with $s \in S$ and $t \notin S$. Let, as above, $c : E \to \mathbb{R}_+$ be an edge capacity function. The *capacity* of an st-cut K is

$$\mathrm{cap}_c(K) = \sum_{e \in K} c(e).$$

The *minimum st-cut problem* is to find an st-cut K with minimum capacity $\mathrm{cap}_c(K)$; such K is called a *minimum cut*. This is a *combinatorial optimization* problem: there is a *finite* number of st-cuts, although this number grows exponentially in the number of vertices of the graph.

Each st-cut gives an upper bound on the maximum flow value, as the following lemma says.

Lemma 1.3 *The following inequality holds*

$$\max\{\mathrm{val}(x) : x \text{ is } st\text{-flow}\} \le \min\{\mathrm{cap}_c(K) : K \text{ is } st\text{-cut}\}.$$

Proof. Let x be an st-flow and let $K = \delta^+(S)$ be an st-cut. From flow conservation in vertices in $S \setminus \{s\}$ we get

$$\begin{aligned} \mathrm{val}(x) &= \textstyle\sum_{v \in S}(\sum_{e \in \delta^+(v)} x(e) - \sum_{e \in \delta^-(v)} x(e)) \\ &= \textstyle\sum_{e \in \delta^+(S)} x(e) - \sum_{e \in \delta^-(S)} x(e) \\ &\le \textstyle\sum_{e \in \delta^+(S)} c(e) \\ &= \mathrm{cap}_c(K) \end{aligned}$$

where the inequality follows from the capacity constraints. By taking the maximum over all st-flows and then taking the minimum over all cuts, the desired inequality is obtained. □

The inequality in Lemma 1.3 can be strengthened: it is actually an equality! This is the classical *max-flow min-cut theorem*, proved by Ford and Fulkerson (1954) (for undirected graphs) and by Dantzig and Fulkerson (for

[2]Linear programming (LP) is to maximize or minimize a linear function subject to linear constraints (linear inequalities and linear equations).

directed graphs, as we consider here). It is considered as one of the most important results in combinatorics and combinatorial optimization. We shall give a short proof of the theorem by using Hoffman's circulation theorem (Theorem 1.1).

Theorem 1.4 (Max-flow min-cut theorem) *For any directed graph D, edge capacity function c, and distinct vertices s, t, the value of a maximum st-flow equals the minimum st-cut capacity, i.e.,*

$$\max\{\text{val}(x) : x \text{ is } st\text{-flow}\} = \min\{\text{cap}_c(K) : K \text{ is } st\text{-cut}\}.$$

Proof. Due to Lemma 1.3 we only need to show that there exists an st-flow with value equal to the minimum cut capacity M. We may assume that D does not contain the edge (t, s) (as a maximum flow exists with zero flow in that edge). Let D' be obtained from D by adding the edge (t, s) (so here the edge is again!). Define $l(t, s) = u(t, s) = M$, and $l(e) = 0$, $u(e) = c(e)$ for each $e \in E$. We shall apply Hoffman's circulation theorem to D', l, u, so consider condition (2). The only interesting case is when $s \in S$, $t \notin S$ (the other case gives a redundant inequality). Then $\sum_{e\in\delta^-(S)} l(e) = M + 0 = M$ while $\sum_{e\in\delta^+(S)} u(e) = \sum_{e\in\delta^+(S)} c(e) = \text{cap}_c(\delta^+(S))$. Thus condition (2) becomes

$$\text{cap}_c(\delta^+(S)) \geq M \quad (S \subseteq V, s \in S, t \notin S).$$

But this condition is satisfied since M is the minimum cut capacity. Thus, by Theorem 1.1 there is a circulation x in D' with $l \leq x \leq u$. So $x(t, s) = l(t, s) = u(t, s) = M$, and therefore the restriction of x to E is an st-flow with value equal to M, and the proof is complete. □

This theorem is an example of a *minmax theorem*: the maximum value of some function taken over some set equals the minimum value of some other function over some set. Another such important minmax theorem is the linear programming duality theorem. Actually, there is a close connection between these two results: one may derive the max-flow min-cut theorem from the duality theorem using the theory of totally unimodular matrices[3]

1.4 Finding a maximum flow and a minimum cut

There are several algorithms for finding a maximum s-flow. Most of them also find a minimum st-cut! We present the classical *Ford-Fulkerson algorithm* (see [9]). It is typically fast, although not theoretically efficient. Today faster

[3] A matrix is called *totally unimodular* (TU) if the determinant of each square submatrix (i.e., a minor) equals -1, 0 or 1.

algorithms exist, even in theory and practice. But the Ford-Fulkerson algorithm is very elegant and easy to explain. Moreover it uses a very important idea, the notion of an augmenting path.

Consider the same setting as in the previos subsection: digraph D, vertices s and t, and a nonnegative capacity function c. We assume that D contains an st-path; otherwise the maximum flow value is zero.

Let x be an st-flow. As in connection with Hoffman's circulation problem we construct an *auxiliary graph* $D_x = (V, E_x)$ where

$$E_x = \{e \in E : x(e) < c(e)\} \cup \{e^{-1} : e \in E,\ x(e) > 0\}.$$

(Recall that $e^{-1} = (v, u)$ when $e = (u, v)$.) An st-path P in D_x is called an *x-augmenting path.* Such a path corresponds to a path in the original graph D: the P-edges that also lie in E are called *forward edges* while the other edges in P correspond to *backward edges* in D. Let P^+ and P^- denote the set of forward and backward edges in D that correspond to P.

The following result is a basic property since it characterizes – constructively – when x is a maximum flow.

Theorem 1.5 *Let x be an st-flow. Then x is a maximum flow if and only if D_x contains no x-augmenting path.*

Proof. Assume first that there is an augmenting path P in D_x. Let ϵ be the minimum of all the following numbers: (i) $c(e) - x(e)$ for each $e \in P^+$, and (ii) $x(e)$ for each $e \in P^-$. So $\epsilon > 0$, by construction of the graph D_x. Then we update the st-flow by adding ϵ to the flow in each edge $e \in P^+$ and subtracting ϵ for each edge in P^-. Let x' be the resulting new flow; this is an st-flow due to the fact the difference between total outflow and total inflow in every vertex $v \neq s, t$ is zero. But $\text{val}(x') = \text{val}(x) + \epsilon$, so we have found an st-flow with larger value. This proves the first part of the theorem.

Next, assume that D_x does not contain an x-augmenting path. Let $S(x)$ denote the set of vertices to which we can find an augmenting sv-path in D_x, and define the cut $K = \delta^+(S(x))$. Then $x(u, v) = c(u, v)$ for each edge $e = (u, v) \in K$ (so $u \in S(x), v \notin S(x)$) ; otherwise we would have $v \in S(x)$. Furthermore, $x(u, v) = 0$ for each edge $e = (u, v)$ with $u \notin S(x), v \in S(x)$ (otherwise $u \in S(x)$). Thus the flow in each edge in the cut K is at its upper capacity while the flow in each edge in the reverse cut $\delta^-(S(x))$ is zero. From this (and flow conservation) it follows that $\text{val}(x) = \text{cap}_c(K)$ (confer the calculations in the proof of Lemma 1.3). So, due to Lemma 1.3, x is a maximum st-flow and $K = \delta^+(S(x))$ is a minimum st-cut. □

From Theorem 1.5 one obtains another proof of the max-flow min-cut theorem, see the Exercises. (This is perhaps the most common proof of this result in the literature.)

The ideas in the proof lead to the following algorithm for finding a maximum flow and a minimum cut.

Ford-Fulkerson max-flow algorithm:

1. Start with the zero flow $x = O$.
2. Look for an x-augmenting path P in D_x.
   a) If such P exists, then find the maximum possible increase ϵ of flow in D along the path corresponding to P (as explained in the proof of Theorem 1.5). Augment (increase) the flow x accordingly.
   b) If no such P exists, then the present x is a maximum flow. Moreover, a minimum st-cut is $\delta^+(S(x))$ where $S(x)$ denote the set of vertices to which we can find an augmenting sv-path in D_x.

One may use a simple procedure (breadth-first search) for finding an x-augmenting path, i.e., an st-path in D_x:

- let $V_0 = \{s\}$, and, iteratively, let V_{i+1} be the vertices in $V \setminus (V_0 \cup \cdots \cup V_i)$ that can be reached by an edge from a vertex in V_i.

Thus, at termination, V_i is the set of vertices with distance i from the source s. This search algorithm requires $O(m)$ steps, where $m = |E|$.

For integral capacities the Ford-Fulkerson algorithm requires at most M iterations, where M is the maximum flow value. This is so because the flow value is increased by at least one in each flow augmentation. It is also easy to see that each flow obtained in the intermediate iterations, in the case of c integral, will have integral edge flows only. This gives the following important *integrality theorem.*

Theorem 1.6 *If the capacity function c is integral (meaning each $c(e)$ is integral), then there is a maximum flow which is integral.*

This theorem has several applications in combinatorics, as we shall see later.

2 Combinatorial matrix theory

Combinatorial matrix theory is an area in matrix theory where one studies combinatorial properties of matrices. A main topic is the study of $(0,1)$-matrices, i.e., matrices with entries consisting of zeros and ones, under different additional constraints on these matrices. Such matrices arise in connection with graphs and, more generally, families of subsets of a finite set, as one often considers in combinatorics. $(0,1)$-matrices also arise from real matrices by replacing its nonzeros by ones. This is done to investigate the role of the *pattern* of nonzeros in different situations.

The main journals for research in combinatorial matrix theory are *Linear Algebra and Its Applications* and *Electronic journal of Linear Algebra*, and some highly recommended monographs on the subject are the book *Combinatorial Matrix Theory* by Richard A. Brualdi and Herbert Ryser [6] and the recent book *Combinatorial Matrix Classes* by Brualdi [4].

We will just present a few ideas and results in this area. First, we introduce the notion of majorization, which is useful in many contexts.

We call a matrix T *nonnegative* if all its entries are nonnegative, and write this as $T \geq O$ where O denotes the all zeros matrix (of suitable size). More generally, $A \leq B$ denotes componentwise ordering ($a_{ij} \leq b_{ij}$ for all i,j) where A and B are matrices of the same size.

2.1 Majorization

The notion of *majorization* is central in matrix theory and its applications. It is a partial order of n-vectors, and we give a brief introduction to the concept.

For $x \in \mathbb{R}^n$ we let $x_{[j]}$ denote the jth largest number among the components of a vector x. If $x, y \in \mathbb{R}^n$ we say that x is *majorized* by y, denoted by $x \preceq y$, provided that

$$\begin{array}{l} \sum_{j=1}^k x_{[j]} \leq \sum_{j=1}^k y_{[j]} \quad (k = 1, 2, \ldots, n-1), \\ \sum_{j=1}^n x_j = \sum_{j=1}^n y_j. \end{array}$$

We refer to Marshall and Olkin's book [15] for a comprehensive study of majorization and its role in many branches of mathematics and applications. Another useful reference here on this is [2]. As an example we have

$$(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}) \preceq (\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0) \preceq (1, 0, \ldots, 0).$$

Majorization turns out to be an underlying structure for several classes of inequalities. One such simple example is the arithmetic-geometric mean inequality

$$(a_1 a_2 \cdots a_n)^{1/n} \leq (1/n)(a_1 + a_2 + \cdots + a_n)$$

which holds for positive numbers $a_1, a_2, \ldots, a_n$. It may be derived, for instance, from the majorization $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}) \preceq (1, 0, \ldots, 0)$ or more directly from the convexity of the logarithm function. See the excellent book [19] for more about this inequality and its generalizations, majorization and the

Cauchy-Schwarz inequality! Another nice illustration of the role of majorization – due to I. Schur – is a majorization order between the diagonal entries $a_{11}, a_{22}, \ldots, a_{nn}$ and the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of a real symmetric (or Hermitian) matrix $A = [a_{ij}]$:

$$(a_{11}, a_{22}, \ldots, a_{nn}) \preceq (\lambda_1, \lambda_2, \ldots, \lambda_n).$$

The corresponding inequalities give, in a certain sense, the best inequalities relating the diagonal entries and the eigenvalues.

Moreover, several interesting inequalities (in geometry, combinaotrics, matrix theory) arise by applying some order-preserving function to a suitable majorization ordering.

The following theorem contains some important classical results concerning majorization, due to Hardy, Littlewood, Polya (1929) and Schur (1923). Recall that a (square) matrix is *doubly stochastic* if it is nonnegative and all row and column sums are equal to one.

Theorem 2.1 *Let $x, y \in \mathbb{R}^n$. Then the following statements are equivalent.*

- (i) $x \preceq y$.
- (ii) *There is a doubly stochastic matrix A such that $x = Ay$.*
- (iii) *The inequality $\sum_{i=1}^n g(x_i) \leq \sum_{i=1}^n g(y_i)$ holds for all convex functions $g : \mathbb{R} \to \mathbb{R}$.*

In particular, this shows that there is a close connection between majorization and doubly stochastic matrices. And this matrix class is closely tied to matching theory; each doubly stochastic matrix corresponds to a fractional perfect matching in a bipartite graph.

For *integral* vectors majorization has a nice characterization in terms of so-called *transfers*. Let $y = (y_1, y_2, \ldots, y_n)$ and assume that $y_i > y_j$ for some pair i, j. Define $y' = (y'_1, y'_2, \ldots, y'_n)$ by $y'_i = y_i - 1$, $y'_j = y_j + 1$ and $y'_k = y_k$ for $k \neq i, j$. We say that y' is obtained from y by a *transfer from i to j*.

Theorem 2.2 *Let x and y be integral vectors of length n. Then $x \preceq y$ if and only if x can be obtained from y by a finite sequence of transfers.*

Majorization plays a rolw in several areas in combinatorics. The *Gale-Ryser theorem* characterizes the existence of a $(0,1)$-matrix with given row and column sum vectors using majorization; we discuss this below; see also R.A. Brualdi's recent book [4].

2.2 Some existence theorems for combinatorial matrices

In this section we discuss a result concerning integral matrices satisfying constraints on line sums (row and column sums) as well as bounds on each entry. The first result is very general and we show how it may be derived from network flow theory.

Theorem 2.3 *Let $R = (r_1, r_2, \dots, r_m)$ and $S = (s_1, s_2, \dots, s_n)$ be nonnegative integral vectors with $\sum_i r_i = \sum_j s_j$, and let $C = [c_{ij}]$ be a nonnegative integral matrix. Then there exists an $m \times n$ (nonnegative) integral matrix $A = [a_{ij}]$ satisfying*

$$\begin{array}{ll} 0 \le a_{ij} \le c_{ij} & (i \le m, j \le n) \\ \sum_{j=1}^n a_{ij} = r_i & (i \le m) \\ \sum_{i=1}^n a_{ij} = s_j & (j \le n) \end{array}$$

if and only if for all $I \subseteq \{1, 2, \dots, m\}$ and $J \subseteq \{1, 2, \dots, n\}$,

$$\sum_{i \in I, j \in J} c_{ij} \ge \sum_{j \in J} s_j - \sum_{i \notin I} r_i. \tag{5}$$

Proof. This follows from Theorem 1.2 by considering the (bipartite) graph D with vertices $u_1, u_2, \dots, u_m$ associated with rows in the matrix C and vertices $v_1, v_2, \dots, v_n$ associated with the columns. The edges are (u_i, v_j) for $i \le m$, $j \le n$. Let $b(u_i) = r_i$ $(i \le m)$ and $b(v_j) = -s_j$ $(j \le n)$. Then an integral flow x in D with $\text{div}_x = b$ and $O \le x \le c$ corresponds to a matrix C with the desired properties. Moreover, condition (3) translates into (5). □

To proceed we need the notion of a conjugate vector. Consider a nonnegative, nonincreasing integral vector $R = (r_1, r_2, \dots, r_m)$, and assume that $r_i \le n$ for each $i \le m$. Define

$$r_k^* = |\{i : r_i \ge k\}| \quad (k \le n)$$

and let $R^* = (r_1^*, r_1^*, \dots, r_n^*)$. The vector R^* is called the *conjugate vector* of R. (Sometimes one augments this vector by some final zeros.) Let $A(R, n)$ denote the $(0, 1)$-matrix of size $m \times n$ with r_i leading ones followed by $n - r_i$ zeros. It is called the *maximal matrix* w.r.t. R. Then the row sum vector of $A(R, n)$ is R and the column sum vector is R^*. Note that R (and R^*) are (integer) partitions of the integer $\tau = \sum_i r_i$. Assume that R is nonincreasing, $r_1 \ge r_2 \ge \cdots \ge r_m$. A diagram with dots, or squares, corresponding to the ones in $A(R, n)$ is called a *Ferrer's diagram* or a *Young diagram* and it is used to study integer partitions.

The following theorem is called the *Gale-Ryser theorem*. It was proved independent of Gale and Ryser. The theorem gives a nice characterization, in terms of majorization, of the existence of a $(0,1)$-matrix with given line sums. The proof we give is due to [12], see also [13], and it relies on majorization theory.

Theorem 2.4 (Gale-Ryser theorem) *Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be nonnegative, nonincreasing integral vectors with $\sum_i r_i = \sum_j s_j$. Then there exists a $m \times n$ $(0,1)$-matrix A with*

$$\begin{array}{l} \sum_{j=1}^{n} a_{ij} = r_i \quad (i \le m) \\ \sum_{i=1}^{n} a_{ij} = s_j \quad (j \le n) \end{array}$$

if and only if

$$S \preceq R^*.$$

Proof. The necessity of the condition follows by looking at the maximal matrix $A(R,n)$. Assume $\mathcal{A}(R,S)$ is nonempty. Since the ones in $A(R,n)$ are left-justified it is clear that any matrix $A = [a_{ij}] \in \mathcal{A}(R,S)$ has at most as many ones in the first k columns as $A(R,n)$ has $(k \le n)$, i.e.

$$\sum_{j=1}^{k} s_j = \sum_{j=1}^{k}\sum_{i=1}^{m} a_{ij} \le \sum_{j=1}^{k}\sum_{i=1}^{m} A(R,n)_{ij} = \sum_{j=1}^{k} r_j^*.$$

So, $S \preceq R^*$.

To prove the converse, assume that $S \preceq R^*$. By Theorem 2.2 S can be obtained from R^* by a finite sequence of transfers, say

$$S = S^{(t)} \preceq S^{(t-1)} \preceq \cdots \preceq S^{(0)} = R^*$$

where $S^{(i)}$ is obtained from $S^{(i-1)}$ by a transfer $(1 \le i \le t)$. Since $\mathcal{A}(R,R^*)$ is nonempty (it contains the maximal matrix $A(R,n)$), we only need to prove the following claim; for then the desired conclusion holds by induction.

Claim: If $\mathcal{A}(R,y)$ is nonempty, and y' is obtained from y by a transfer from i to j, then also $\mathcal{A}(R,y')$ is nonempty.

Proof of Claim: Let y and y' be as in the claim, and assume $A = [a_{ij}] \in \mathcal{A}(R,y)$. Then $y_i > y_j$, and therefore there is a row k in A where $a_{ki} = 1$ and $a_{kj} = 0$ (as y_i and y_j are column sums in A). Let $A' = [a'_{pq}]$ be obtained from A by letting $a'_{ki} = 0$ and $a'_{kj} = 1$ while A' agrees with A in all other positions. Then $A' \in \mathcal{A}(R,y')$.

This proves the claim, and therefore the theorem. ∎

There are other proofs of this theorem, and the original proofs by Gale and Ryser (independently) were constructive. In particular, *Ryser's algorithm* for constructing a matrix in $\mathcal{A}(R,S)$ when $S \preceq R^*$ is interesting. The algorithm is simple and it produces a "canonical matrix" in $\mathcal{A}(R,S)$ with specific properties (see [4]).

2.3 The matrix class $\mathcal{A}(R,S)$

Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be given nonnegative integral vectors with $\sum_i r_i = \sum_j s_j$. Define $\mathcal{A}(R,S)$ to be the class (set) of $(0,1)$-matrices with row sum vector R and column sum vector S. The row sum vector of A is the vector whose i'th component is the sum of the entries in the ith row in A; the column sum vector is defined similarly. A vector $z = (z_1, z_2, \ldots, z_p)$ is called *nonincreasing* if $z_1 \geq z_2 \geq \cdots \geq z_p$. We shall assume that both R and S are nonincreasing. This can be done since there is an obvious bijection between a class $\mathcal{A}(R,S)$ and another class $\mathcal{A}(R',S')$ where R' is a permutation of R and S' is a permutation of S: simply permute rows and columns similarly. By the Gale-Ryser theorem the class $\mathcal{A}(R,S)$ is nonempty if and only if $S \preceq R^*$.

Assume now that R and S are nonincreasing. Define an $(m+1)\times(n+1)$ matrix $T = [t_{kl}]$ whose entries are t_{kl} $(0 \leq k \leq m,\ 0 \leq l \leq n)$ given by

$$t_{kl} = kl + \sum_{i=k+1}^{m} r_i - \sum_{j=1}^{l} s_j. \tag{6}$$

The matrix T is called the *structure matrix* associated with the class $\mathcal{A}(R,S)$. As the name indicates this matrix reveals strutural properties of matrices in $\mathcal{A}(R,S)$. The structure matrix was introduced by Ford and Fulkerson who proved the following result.

Theorem 2.5 *$\mathcal{A}(R,S)$ is nonempty if and only if T is nonnegative.*

Proof. Consider the special case of Theorem 2.3 where $c_{ij} = 1$ for all i,j, so we consider $(0,1)$-matrices with row sum vector R and column sum vector S. By the theorem such a matrix exists if and only if (5) holds. Our goal is to show that this condition is equivalent to $T \geq O$. Clearly, (5) is equivalent to

$$(*_1)\quad \textstyle\sum_{i\in I, j\in J} c_{ij} \geq \sum_{j\in J} s_j - \sum_{i\notin I} r_i \quad (|I| = k, |J| = l)$$

for all $0 \leq k \leq m, 0 \leq l \leq n$. In $(*_1)$ $\sum_{i\in I, j\in J} c_{ij} = kl$. Moreover, the maximum value of the right-hand side in $(*_1)$, when $|I| = k, |J| = l$, is obtained for $J = \{1, 2, \ldots, l\}$ and $I = \{1, 2, \ldots, k\}$ since R and S are assumed nonincreasing. It follows that (5) is equivalent to

$$kl \geq \sum_{j=1}^{l} s_j - \sum_{i=k+1}^{m} r_i \quad (0 \leq k \leq m, 0 \leq l \leq n)$$

which means that the structure matrix T is nonnegative (see (6), and we are done. □

There is an important aspect of different characterizations of the nonemptyness of the class $\mathcal{A}(R, S)$. The direct application of Theorem 2.3 gives a condition with many inequalities, one for each $I \subseteq \{1, 2, \ldots, m\}$ and $J \subseteq \{1, 2, \ldots, n\}$. There are 2^{m+n} such inequalities, so the number grows exponentially fast. Theorem 2.5 improves on this and contains $(m+1)(n+1)$ inequalities, one for each entry in the structure matrix. This reduction, as we saw in the proof, was done by eliminating redundant inequalities using the fact that R and S were nonincreasing. Finally, the winner in this contest is the Gale-Ryser theorem which gives a characterization in terms of only $n - 1$ inequalities!

The structure matrix may be given a combinatorial interpretation as we discuss next. The number of zeros resp. ones of a $(0, 1)$-matrix C is denoted by $N_0(C)$ resp. $N_1(C)$. Now, consider a $(0, 1)$-matrix A of size $m \times n$ and partitioned as follows

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{11} is of size $k \times l$. Assume that $A \in \mathcal{A}(R, S)$. Then

$$\begin{aligned} t_{kl} &= kl + \textstyle\sum_{i=k+1}^{m} r_i - \sum_{j=1}^{l} s_j \\ &= (N_0(A_{11}) + N_1(A_{11})) + (N_1(A_{21}) + N_1(A_{22})) - (N_1(A_{11}) + N_1(A_{21})) \\ &= N_0(A_{11}) + N_1(A_{22}). \end{aligned}$$

Thus, t_{kl} counts something: it is the number of zeros in A_{11} (which has size $k \times l$) plus the number of ones in A_{22}. This shows (some of) the combinatorial meaning of the structure matrix T. Note that, from this interpretation, it is clear that if $\mathcal{A}(R, S)$ is nonempty, then T is nonnegative. The converse implication, however, is much more difficult to prove.

The matrix T has a number of interesting properties. We mention some of these. The entries of T are all determined by the entries in the first (actually, zero'th) row and column as we have

$$
\begin{array}{lll}
t_{00} & = \tau = \sum_i r_i = \sum_j s_j & \\
t_{0l} & = \sum_{j=l+1}^{n} s_j & (l = 0, 1, \ldots, n) \\
t_{k0} & = \sum_{j=k+1}^{m} r_i & (k = 0, 1, \ldots, m) \\
t_{k+1,l+1} & = t_{k+1,l} + t_{k,l+1} - t_{kl} & (k = 0, 1, \ldots, m-1, l = 0, 1, \ldots, n-1).
\end{array}
$$

The structure matrix T does indeed reveal a lot about the structure of matrices in $\mathcal{A}(R, S)$. For instance, assume that $t_{kl} = 0$ some k, l with $k, l \geq 1$. By the combinatorial interpretation $t_{kl} = N_0(A_{11}) + N_1(A_{22})$ this means that every matrix in $A \in \mathcal{A}(R, S)$ satisfies

$$
\begin{array}{ll}
a_{ij} = 1 & (i \leq k, j \leq l) \\
a_{ij} = 0 & (i \geq k+1, j \geq l+1).
\end{array}
$$

The structure matrix plays a significant role in deeper investigations of the class $\mathcal{A}(R, S)$, see [4], [5].

2.4 Doubly stochastic matrices

A (real) $n \times n$ matrix A is called *doubly stochastic* if it is nonnegative and all its row and column sums are one. These matrices arise in connection with stochastic processes, optimization (the assignment problem), majorization, combinatorics etc. An example is

$$
A = \begin{bmatrix} 0.4 & 0.2 & 0.4 \\ 0.6 & 0.3 & 0.1 \\ 0 & 0.5 & 0.5 \end{bmatrix}
$$

Let Ω_n be the set of all $n \times n$ doubly stochastic matrices. This set, or matrix class, has been studied a lot in matrix theory. In 1929 Hardy, Littlewood and Pólya proved that majorization is closely related to doubly stochastic matrices: $y \preceq x$ if and only if there is a doubly stochastic matrix A such that $y = Ax$, see Theorem 2.1. Thus, $y \preceq x$ if and only if y is the image of x under a certain linear transformation, a doubly stochastic map.

There is also a close connection between Ω_n and certain combinatorial objects. Permutations are important in many areas (group theory, combinatorics, applications in sequencing etc.). A *permutation* σ is simply a reordering of the integers $1, 2, \ldots, n$, and we write this as a vector $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$. Let S_n be the set of all n-permutations. For instance, $\sigma = (3, 1, 4, 2) \in S_4$. The set S_n is a group, called the *symmetric group*, where the group operation is (function) composition. Here one may think of a permutation as a bijection (a function which is one-to-one and onto) on a set of n elements, e.g., $\{1, 2, \ldots, n\}$. The function, also denoted by σ is then $\sigma(i) = \sigma_i$.

Now, permutations may be represented in other ways as well, also by matrices, so-called permutation matrices. A *permutation matrix* is a $(0, 1)$-matrix with exactly one 1 in every row and in every column, e.g.,

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

A permutation $\sigma \in S_n$ may be represented by the matrix $P = [p_{ij}] \in \mathbb{R}^{n \times n}$ where $p_{ij} = 1$ when $j = \sigma(i)$ and $p_{ij} = 0$ otherwise. The permutation matrix just shown corresponds to the permutation $\sigma = (3, 1, 2)$. This is a permutation matrix, and this defines a bijection between S_n and the set $\mathcal{P}_n$ of all permutation matrices of order n. (One may also use the transpose P^T to represent a permutation; if this is done function composition of permutations corresponds in the natural way to matrix multiplication.)

Now, note that each permutation matrix is also a doubly stochastic matrix. In fact, the permutation matrices are precisely the integral doubly stochastic matrices. A clasical theorem of Birkhoff (1946) and von Neumann (1953) gives an elegant connection between these two matrix classes. Recall that the *convex hull* of a set S (in a vector space), denoted by $\text{conv}(S)$, is the set of all convex combinations of points in S; it is the smallest (inclusionwise) convex set containing S.

Theorem 2.6 (Birkhoff-von Neumann theorem) *The set Ω_n of all doubly stochastic matrices of order n is the convex hull of all permutation matrices of order n,*

$$\Omega_n = \text{conv}(\mathcal{P}_n).$$

Actually, $\mathcal{P}_n$ is the set of vertices of Ω_n. So every doubly stochastic matrix may be written as a convex combination of permutation matrices.

Proof. We give a proof using polyhedral theory. Since a polytope is the convex hull of its vertices, it suffices to prove that the vertices of Ω_n are the permutation vertices.

First, we note that every permutation matrix is a vertex of Ω_n (Exercise), and that these are precisely the integral matrices in Ω_n. Next, let $A = [a_{ij}]$ be a vertex of Ω_n which is non-integral, so there exists (i_1, j_1) with $0 < a_{i_1 j_1} < 1$. Since the j_1'th column sum is 1 in A, there is an $i_2 \neq i_1$ with $0 < a_{i2 j_1} < 1$. And since the i_2'th row sum is 1 in A, there is an $j_2 \neq j_1$

with $0 < a_{i_2 j_2} < 1$. Continuing like this we must eventually obtain a cycle C with vertices $i_1, j_1, i_2, j_2, \ldots, i_k, j_k, i_1$ in the bipartite graph whose vertices correspond to rows and columns in A and the edges correspond to positions (or entries) in A. This is an even cycle with edges, say, $e_1, e_2, \ldots, e_{2t}$, and the corresponding entries in A lie in the open interval $\langle 0, 1 \rangle$. Let V be the $n \times n$ matrix with a 1 in the positions corresponding to edges e_i with $i \leq 2t$ and i odd, and -1 in the positions corresponding to edges e_i with $i \leq 2t$ and i even. Then, for suitabley small $\epsilon > 0$, the matrices $A^1 = A + \epsilon V$ and $a^2 = A - \epsilon V$ both lie in Ω_n. But since $A = (1/2)(A^1 + A^2)$, this contradicts that A is an extreme point. Thus, there are only integral extreme points, and these are the permutation matrices. □

A detailed treatment of the *Birkhoff polytope* Ω_n may be found in [4].

Exercises

1. Let x be a flow in a graph D. Show that $\sum_{v \in V} \text{div}_x(v) = 0$.

2. Let x be a circulation in a graph $D = (V, E)$ and let $S \subseteq V$. Prove that $\sum_{e \in \delta^-(S)} x(e) = \sum_{e \in \delta^+(S)} x(e)$. (Hint: sum the flow balance equations for vertices in S.)

3. Consider the problem treated in Hoffman's circulation theorem (Theorem 1.1): decide if a circulation x satisfying $l \leq x \leq u$ exists and, if so, find one. Show that this problem may be transformed into a flow problem with zero lower bounds, but with given divergence. Hint: apply the transformation $x'(e) = x(e) - l(e)$ $(e \in E)$.

4. Show that when x is an st-flow, then $\text{val}(x)$ is equal to the total inflow to the sink t.

5. Explain why the Extreme Value Theorem gives the existence of a maximum flow. Also explain how this following from linear programming theory.

6. Use Theorem 1.5 to give a proof of the max-flow min-cut theorem. Hint: consider a maximum flow.

7. Choose an example of D, s, t, c and find a maximum flow and minimum cut using the Ford-Fulkerson algorithm.

8. How can we find a flow x with given divergence, say $\text{div}_x = b$, and satisfying capacity constraints $O \leq x \leq c$? This is an important problem and it can be solved by transforming it into a maximum flow problem (for which several extremely fast algorithms exist). The construction is very similar to the one we gave in the proof of Theorem 1.2. Let $D = (V, E)$ be the given graph. Define $V^- = \{v \in V : b(v) < 0\}$ and $V^+ = \{v \in V : b(v) > 0\}$. Construct a graph $D' = (V', E')$ with vertex set $V' = V \cup \{s, t\}$, so we add *two* new vertices s and t. Let the edge set of D' consist of (i) each edge $e \in E$, (ii) an edge (s, v) for each $v \in V^+$, and (iii) an edge (v, t) for each $v \in V^-$. Define a capacity function $c' : E' \to \mathbb{R}$ by $c'(s, v) = b(v)$ for each $v \in V^+$, $c'(v, s) = -b(v)$ for each $v \in V^-$, and $c'(e) = c(e)$ for each $e \in E$.

   The questions: (a) Show that the maximum value of an st-flow is at most $M := \sum_{v \in V^+} b(v)$. (b) Show that a flow x in D satisfying $\text{div}_x = b$ and $O \leq x \leq c$ exists if and only if the value of a maximum st-flow in D' equals M (defined in (a)). (c) How do you find the desired x from this maximum flow?

9. Fill in the details of the proof of Theorem 1.6 (the integrality theorem).

10. Consider the max-flow problem in the special case where $c(e) = 1$ for each $e \in E$ (unit capacities). Choose and example of D, s and t. Solve the max-flow problem. What can be said about the structure of your max-flow x? Prove, in general, that a max-flow may be represented by a set of *edge-disjoint st*-paths (i.e., pairwise disjoint) and where there is unit flow on each of these paths.

11. Prove *Menger's theorem: the maximum number of edge-disjoint st-paths is equal to the minimum cardinality of an st-cut.* This is a classical minmax theorem in graph theory. Hint: consider the previous exercise.

12. The *matrix rounding problem* is the following problem. Given a real $m \times n$ matrix $A = [a_{ij}]$ with row sums $r_i = \sum_{j=1}^{n} a_{ij}$ $(i \leq m)$ and column sums $s_j = \sum_{i=1}^{n} a_{ij}$ $(j \leq n)$, round each entry a_{ij} to either $\lfloor a_{ij} \rfloor$ or $\lceil a_{ij} \rceil$ and also round each r_i and s_j either up or down, such that the new matrix $\bar{A}$ has row and column sums equal to the corresponding rounded sums. For instance, consider

$$A = \begin{bmatrix} 4.2 & 3.5 & 2.6 \\ 1.1 & 2.1 & 8.3 \\ 6.5 & 3.9 & 1.2 \end{bmatrix}.$$

Here $(r_1, r_2, r_3) = (10.3, 11.5, 11.6)$ and $(s_1, s_2, s_3) = (11.8, 9.5, 12.1)$. Then we might start by rounding a_{11}, a_{12}, a_{13} and r_1 to 4, 4, 3 and 11, respectively. Go on, and try to solve the problem. The difficulty is that we have to get the "right" column sums as well. The general matrix rounding problem may be represented as a flow problem in the following graph. Let $G = (V, E)$ be a graph with vertex set $V = \{s, t, u_1, \ldots, u_m, v_1, \ldots, v_n\}$ and the following edges (i) (s, u_i) for $i \leq m$, (ii) (v_j, t) for $j \leq n$, (iii) (u_i, v_j) for $i \leq m, j \leq n$, and (iv) the single edge (t, s). The idea is to represent the matrix entry $\bar{a}_{ij}$ by the flow $x(e)$ in the edge $e = (u_i, v_j)$. Moreover, the ith row sum corresponds to the flow in the edge (s, u_i), and the jth column sum corresponds to the flow in the edge (v_j, t). Draw the graph for the specific example above. Your task is to define a lower bound $l(e)$ and an upper bound $u(e)$ on the flow in each edge e such that the matrix rounding problem becomes that of finding a circulation x satisfying $l \leq x \leq u$.

13. Let P be a polytope contained in the unit cube, so each $x \in P$ satisfies $0 \leq x_j \leq 1$ $(j \leq n)$. Prove that each $(0, 1)$-vector in P is a vertex of P.

14. Show that every permutation matrix is a vertex of Ω_n and that these are precisely the integral matrices in Ω_n. Hint: use the previous exercise.

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