UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in INF-MAT 5360 — Mathematical optimization

Day of examination: December 2., 2010

Examination hours: 14.30 – 18.30

This problem set consists of 4 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

There are 12 questions with about the same weight.

Problem 1

1a

Let $C \subseteq \mathbb{R}^n$ be a convex set and consider a line $L = \{x \in \mathbb{R}^n : x = a + tr,\ t \in \mathbb{R}\}$ where $a, r \in \mathbb{R}^n$ are given vectors. Is $C \cap L$ a convex set? Depending on your answer, give a proof or a counterexample.

1b

Let $a, b \in \mathbb{R}^n$ and consider the set

$$S = \{x \in \mathbb{R}^n : \|x - a\| \leq \|x - b\|\}$$

where $\|z\| = \sqrt{z^T z}$ is the Euclidean norm of a vector z. Show that S is a halfspace. (Hint: work on the inequalities in the definition of S). Give an example in the plane, i.e., when $n = 2$.

Problem 2

2a

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $\alpha \in \mathbb{R}$. Show that the set

$$K = \{x \in \mathbb{R}^n : f(x) \leq \alpha\}$$

is convex.

(Continued on page 2.)

2b

Let $x \in \mathbb{R}^n$ be a convex combination of the vectors $z_1, z_2, \ldots, z_k \in \mathbb{R}^n$, and let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Show that

$$f(x) \leq \max\{f(z_1), \ldots, f(z_k)\}.$$

(Hint: Jensen's inequality)

2c

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a polyhedron (so A is a real $m \times n$ matrix and $b \in \mathbb{R}^m$). Show that the recession cone of P is given by

$$\text{rec}(P) = \{z \in \mathbb{R}^n : Az \leq O\}$$

where O is the zero vector.

2d

Let $C \subset \mathbb{R}^3$ be the unit cube, i.e.,

$$C = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 \leq x_i \leq 1 \ \ (i \leq 3)\}$$

Determine, with reference to general theory, each face F of C such that $\dim(F) = 1$ and F contains the point $(1, 1, 1)$.

Problem 3

3a

Give an example of a 2×2 matrix $B = [b_{ij}]$ where (a) $b_{ij} \in \{-1, 0, 1\}$ for $1 \leq i, j \leq 2$ and (b) B is *not* totally unimodular. Also, give a proof of the following fact: the node-edge incidence matrix of a directed graph is totally unimodular. (This is a result in the lecture notes.)

Problem 4

Let $F(G) = \{x \in \mathrm{I\!R}^E : Ax \leq b\}$ be the forest polytope associated with the undirected graph $G = (V, E)$ in Figure 1.a). An inequality of type $x_e \geq 0$ $(e \in E)$ is said to be a *trivial inequality*.

4a

Let $\mathcal{A}$ be a separation oracle for $F(G)$ and let $\hat{x} \in \mathrm{I\!R}^E$ be the point indicated in the picture (i.e. $x_{13} = 2/3$, $x_{12} = x_{14} = 1/3$, $x_{23} = 0$, $x_{24} = x_{34} = 1$). If the input to $\mathcal{A}$ is $\hat{x}$, what will it be its output?

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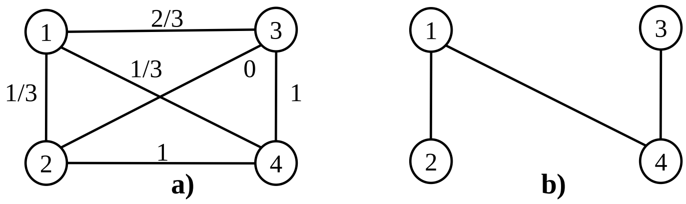

Figure 1:

4b

Consider the spanning tree H of G given in Figure 1.b). The incidence vector of H is $x^H_{12} = x^H_{14} = x^H_{34} = 1$, $x^H_{13} = x^H_{23} = x^H_{24} = 0$. Show that x^H is a vertex of $F(G)$, without using trivial inequalities in your proof.

Problem 5

Let $G = (V, E)$ be an undirected simple graph (no loops, no multiple edges). A *stable set* of vertices is a set $S \subseteq V$ of pairwise non-adjacent vertices of G, i.e. for all $i, j \in S$ we have $[i, j] \notin E$. For example, in Figure 2, the set $S = \{2, 3, 5\}$ is a stable set.

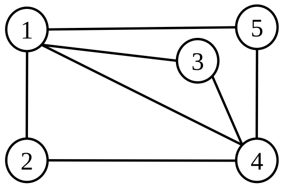

Figure 2:

Let $Q(G) \subseteq \{0, 1\}^V$ be the set of incidence vectors of stable sets of G. It is not difficult to see that a $(0, 1)$-vector $x \in \mathbb{R}^V$ lies in $Q(G)$ if and only if it satisfies $x_u + x_v \leq 1$ for all $[u, v] \in E$. In other words, the polyhedron $P(G) = \{x \in \mathbb{R}^V : x_u \geq 0 \text{ for all } u \in V,\ x_u + x_v \leq 1 \text{ for all } [u, v] \in E\}$ is a *formulation* of $Q(G)$.

Consider now three distinct and pairwise adjacent vertices v, w, z of G, i.e. $\{[v, w], [w, z], [v, z]\} \subseteq E$.

5a

Show that the inequality $x_v + x_w + x_z \leq 1$ is valid for the convex hull of $Q(G)$.

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5b

Show that the *clique* inequality $x_v + x_w + x_z \leq 1$ is not valid for $P(G)$.

Problem 6

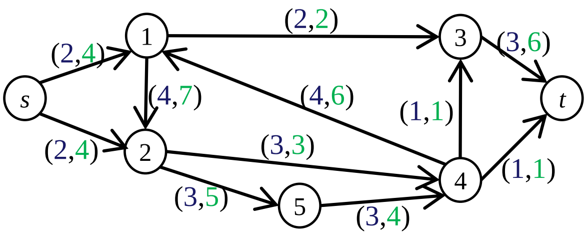

Figure 3:

Consider the graph in Figure 3 where flow x_e and capacity c_e are shown next to each edge e (in this order).

6a

Show that the given flow x is a maximum st-flow.

Good luck!