UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in:	MAT-INF4110/MAT-INF9110 — Mathematical Optimization
Day of examination:	Monday, December 2, 2013
Examination hours:	09.00 – 13.00

This problem set consists of 3 pages.

Appendices:	None
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

There are 11 questions with about the same weight, except that 1b has double weight.

Problem 1

1a

Let x be a flow in a (directed) graph $D = (V, E)$. How is the divergence Div_x of x defined? Show that $\sum_{v \in V} \text{Div}_x(v) = 0$.

1b

State Hoffman's circulation theorem, and prove it (at least the main steps).

Consider the (directed) graph $D = (V, E)$ shown in Figure 1. Let the functions l and u from E into $\mathbb{R}$ be as indicated in the figure, where the pair $(l(e), u(e))$ is shown along each edge e.

1c

Prove that there is no circulation x in D which satisfies $l \le x \le u$.

1d

Explain the Ford-Fulkerson max-flow algorithm. In this connection you should also explain the notion of an x-augmenting path and the graph D_x.

Problem 2

2a

Let $m = 6$, $n = 8$, $R = (5, 5, 5, 3, 2, 2)$ and $S = (5, 5, 5, 5, 1, 1, 0, 0)$. Does there exist a $(0, 1)$-matrix of size 6×8 such that A has row sum vector R and column sum vector S? Explain.

(Continued on page 2.)

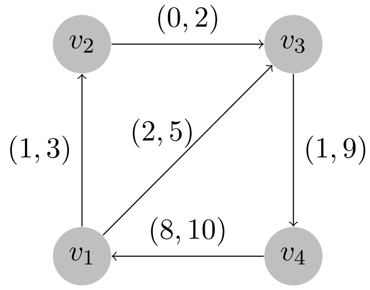

Figure 1: The graph D, showing $(l(e), u(e))$.

Consider the matrix

$$A = \begin{bmatrix} 0 & 0.6 & 0.4 \\ 0.7 & 0 & 0.3 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}.$$

2b

Show that A can be written as a convex combination $\sum_k \lambda_k P_k$ of permutation matrices P_k by finding such P_k's and λ_k.

Let $x^1, x^2, \ldots, x^{47}$ be 47 distinct points (vectors) in $\mathbb{R}^{10}$ and let P be the convex hull of these 47 points. Assume that $z \in P$.

2c

Explain why z can written as a convex combination of at most 11 of these 47 points. State (but not prove) the theorem you use here. Next, let

$$R = \{x \in \mathbb{R}^3 : x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_1+x_2 \leq 1,\ x_1+x_3 \leq 1,\ x_2+x_3 \leq 1\}.$$

Show that $y = (1/2, 1/2, 1/2)$ is an extreme point of R.

Problem 3

Let S be the following set of 5 points in $\mathbb{R}^3$:

$$S = \{(0,0,0), (1,0,0), (0,1,0), (1,1,0), (1,1,1)\}.$$

Let $P = \text{conv}(S)$.

3a

Determine each face F of P that contains $(1,1,1)$. You shall specify each such F by giving its extreme points and determine the dimension of F.

Let K be a polytope in $\mathbb{R}^n$ and let H be a halfspace in $\mathbb{R}^n$.

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3b

Define $L := K \cap H$. Show that L is a polytope (you may refer to a general theorem in this proof). Let x be an extreme point of K and assume that $x \in H$. Show that x is also an extreme point of L. Finally, give an example (for n being 1 or 2) that L may have other extreme points that those of K.

Problem 4

4a

Let T be a tree (i.e., a connected graph with no cycles). Let n be the number of vertices of T, and m its number of edges. Prove that $m = n - 1$.

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

4b

Is A totally unimodular (TU)? Find an integral vector $b \in \mathbb{R}^4$ such that the polyhedron

$$P = \{x \in \mathbb{R}^4 : Ax \le b,\ x \ge O\}$$

is *not* integral. Explain your answers.

Good luck!