Weights in the barycentric Lagrange formula

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These notes derive the weights in the barycentric Lagrange formula when the interpolation points are Chebyshev points.

Let $p \in \pi_n$ be the polynomial interpolant, of degree $< n$, to $f_1, \ldots, f_n \in \mathbb{R}$ at the distinct points $x_1, \ldots, x_n \in \mathbb{R}$. The barycentric Lagrange formula for p is

$$p(x) = \sum_{i=1}^{n} \frac{w_i}{x - x_i} f_i \Big/ \sum_{i=1}^{n} \frac{w_i}{x - x_i},$$

where the weights are

$$w_i = 1 \Big/ \prod_{\substack{j=1 \\ j \neq i}}^{n} (x_i - x_j).$$

Note that the formula for $p(x)$ is unchanged if the weights are scaled by a constant.

Theorem 1 *If $x_1, \ldots, x_n$ are the Chebyshev points of the first kind:*

$$x_i = \cos\left(\frac{2i-1}{n}\frac{\pi}{2}\right), \qquad i = 1, \ldots, n,$$

then

$$w_i = (-1)^i \sin\left(\frac{2i-1}{n}\frac{\pi}{2}\right).$$

Proof. Using the sum-to-product identity,

$$\cos A - \cos B = -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right),$$

we find

$$x_i - x_j = -2 s_{i+j-1} s_{i-j},$$

where

$$s_k := \sin\left(\frac{k}{n}\frac{\pi}{2}\right).$$

It follows that

$$\prod_{j \neq i} (x_i - x_j) = (-2)^{n-1} ab,$$

where

$$a = \prod_{j \neq i} s_{i+j-1}, \qquad b = \prod_{j \neq i} s_{i-j}.$$

Then

$$a = \frac{s_i s_{i+1} \cdots s_{i+n-1}}{s_{2i-1}},$$

and

$$b = s_{i-1} \cdots s_1 s_{-1} \cdots s_{i-n},$$

and since $s_{k+2n} = -s_k$,

$$b = (-1)^{n-i} s_{i-1} \cdots s_1 s_{2n-1} \cdots s_{i+n},$$

and so

$$ab = \frac{(-1)^{n-i} C_n}{s_{2i-1}},$$

where

$$C_n = s_1 \cdots s_{2n+1},$$

which is independent of i, and

$$w_i = \frac{(-1)^{i-1}}{2^{n-1} C_n} s_{2i-1}.$$

□

Theorem 2 *If $x_1, \ldots, x_n$ are the Chebyshev points of the second kind:*

$$x_i = \cos\left(\frac{i-1}{n-1}\pi\right), \qquad i = 1, \ldots, n,$$

then $w_i = (-1)^i \delta_i$, where $\delta_i = 1$ for $i = 2, \ldots, n-1$ and $\delta_1 = \delta_n = 1/2$.

Proof. Using the sum-to-product identity again, and with

$$s_k := \sin\left(\frac{k}{n-1}\frac{\pi}{2}\right).$$

we find

$$x_i - x_j = -2s_{i+j-2}s_{i-j},$$

and so

$$\prod_{j\neq i}(x_i - x_j) = (-2)^{n-1}ab,$$

where

$$a = \prod_{j\neq i} s_{i+j-2}, \qquad b = \prod_{j\neq i} s_{i-j}.$$

If $2 \leq i \leq n-1$, then

$$a = \frac{s_{i-1}s_i \cdots s_{i+n-2}}{s_{2i-2}},$$

and

$$b = s_{i-1}\cdots s_1 s_{-1}\cdots s_{i-n},$$

but since now $s_{k+2n-2} = -s_k$,

$$b = (-1)^{n-i}s_{i-1}\cdots s_1 s_{2n-3}\cdots s_{i+n-2}.$$

Therefore,

$$ab = (-1)^{n-i}C_i D_n, \tag{1}$$

where

$$C_i = \frac{s_{i-1}s_{i+n-2}}{s_{2i-2}},$$

and

$$D_n = s_1 s_2 \cdots s_{2n-3}.$$

By the double angle formula for sines, we have

$$s_{2i-2} = s_{2(i-1)} = 2s_{i-1}c_{i-1},$$

where

$$c_k := \cos\left(\frac{k}{n-1}\frac{\pi}{2}\right).$$

Since further $s_{i+n-2} = c_{i-1}$, we find

$$C_i = 1/2.$$

This gives the formula for w_i for $i = 2, \ldots, n-1$. It remains to consider the cases $i = 1$ and $i = n$. In the case $i = 1$,

$$a = \prod_{j=2}^{n} s_{j-1} = s_1 \cdots s_{n-1},$$

and

$$b = \prod_{j=2}^{n} s_{1-j} = (-1)^{n-1} \prod_{j=2}^{n} s_{2n-j-1} = (-1)^{n-1} s_{n-1} \cdots s_{2n-3},$$

and so

$$ab = (-1)^{n-1} s_{n-1} D_n = (-1)^{n-1} D_n,$$

which agrees with (1) if $C_1 = 1$.

In the case $i = n$,

$$a = \prod_{j=1}^{n-1} s_{n+j-2} = s_{n-1} \cdots s_{2n-3},$$

and

$$b = \prod_{j=1}^{n-1} s_{n-j} = s_1 \cdots s_{n-1},$$

and so

$$ab = s_{n-1} D_n = D_n,$$

which again agrees with (1) if $C_n = 1$.

$\square$