Hermite interpolation

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These notes extend the notion of Lagrange interpolation to Hermite interpolation. We study iterative interpolation and the Newton form.

1 Hermite interpolation

Suppose that $x_0, x_1, \ldots, x_n$ are distinct points in $[a, b]$ and that f is a function that has derivatives of orders $0, 1, \ldots, r_i$, for each $i = 0, 1, \ldots, n$.

Theorem 1 *With*

$$N = n + \sum_{i=0}^{n} r_i,$$

there is a unique polynomial $p \in \pi_N$ *such that*

$$p^{(k)}(x_i) = f^{(k)}(x_i), \qquad i = 0, 1, \ldots, n, \quad k = 0, 1, \ldots, r_i. \tag{1}$$

Proof. Any $p \in \pi_N$ can be expressed uniquely as

$$p(x) = \sum_{j=0}^{N} c_j x^j,$$

and its k-th derivative is

$$p^{(k)}(x) = \sum_{j=k}^{N} \frac{j!}{(j-k)!} c_j x^{j-k}.$$

The interpolation conditions (1) are then

$$\sum_{j=k}^{N} \frac{j!}{(j-k)!} c_j x_i^{j-k} = f^{(k)}(x_i), \qquad i = 0,1,\ldots,n, \quad k = 0,1,\ldots,r_i.$$

which can be expressed as the linear system

$$Mc = f, \tag{2}$$

where $c = (c_0, c_1, \ldots, c_N)^T$ and

$$M = \begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 & \cdots & x_0^N \\ & x_0 & 2x_0 & 3x_0^2 & \cdots & Nx_0^{N-1} \\ & & & & & \vdots \\ 1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^N \\ & x_1 & 2x_1 & 3x_1^2 & \cdots & Nx_1^{N-1} \\ & & & & & \vdots \end{bmatrix}, \quad f = \begin{bmatrix} f(x_0) \\ f'(x_0) \\ \vdots \\ f(x_1) \\ f'(x_1) \\ \vdots \end{bmatrix},$$

and it is sufficient to show that M is non-singular. To demonstrate this suppose that c satisfies the homogeneous equation $Mc = 0$. Then the polynomial

$$q(x) = \sum_{j=0}^{N} c_j x^j$$

satisfies the conditions

$$q^{(k)}(x_i) = 0, \qquad i = 0,1,\ldots,n, \quad k = 0,1,\ldots,r_i.$$

Then $q \in \pi_N$ and has at least $N+1$ roots, counting multiplicities, and, similar to the Lagrange case, by the fundamental theorem of algebra, $q = 0$. Hence $c = 0$ and M is indeed non-singular. $\square$

2 Iterative interpolation

One way of finding the Hermite interpolant $p \in \pi_N$ is through the same iterative procedure we looked at in the first lecture. First, observe that if $n = 0$, the interpolant is the Taylor polynomial,

$$p(x) = \sum_{j=0}^{r_0} f^{(j)}(x_0) \frac{(x-x_0)^j}{j!}. \tag{3}$$

Otherwise $n \geq 1$, and suppose that $q \in \pi_{N-1}$ satisfies the conditions

$$q^{(k)}(x_i) = f^{(k)}(x_i), \qquad i = 0, 1, \ldots, n-1, \quad k = 0, 1, \ldots, r_i,$$
$$q^{(k)}(x_n) = f^{(k)}(x_n), \qquad k = 0, 1, \ldots, r_n - 1,$$

where we understand the second condition to be 'empty' if $r_n = 0$, and similarly suppose that $r \in \pi_{N-1}$ satisfies

$$r^{(k)}(x_0) = f^{(k)}(x_0), \qquad k = 0, 1, \ldots, r_0 - 1,$$
$$r^{(k)}(x_i) = f^{(k)}(x_i), \qquad i = 1, 2, \ldots, n, \quad k = 0, 1, \ldots, r_i.$$

Theorem 2 *The polynomial*

$$p(x) := \frac{x_n - x}{x_n - x_0} q(x) + \frac{x - x_0}{x_n - x_0} r(x) \tag{4}$$

is the polynomial in π_N that solves the Hermite interpolation problem (1).

Proof. By the Leibniz rule, the k-th derivative of p in (4) is

$$p^{(k)}(x) = \frac{x_n - x}{x_n - x_0} q^{(k)}(x) + \frac{x - x_0}{x_n - x_0} r^{(k)}(x) + k \frac{r^{(k-1)}(x) - q^{(k-1)}(x)}{x_n - x_0},$$

and it follows that

$$p^{(k)}(x_i) = \frac{x_n - x_i}{x_n - x_0} q^{(k)}(x_i) + \frac{x_i - x_0}{x_n - x_0} r^{(k)}(x_i),$$

for all $i = 0, 1, 2, \ldots, n$ and $k = 0, 1, \ldots, r_i$. Thus,

$$p^{(k)}(x_0) = q^{(k)}(x_0) = f^{(k)}(x_0), \quad k = 0, 1, \ldots, r_0,$$

and,

$$p^{(k)}(x_n) = r^{(k)}(x_n) = f^{(k)}(x_n), \quad k = 0, 1, \ldots, r_n,$$

and, for $i = 1, \ldots, n-1$,

$$p^{(k)}(x_i) = \frac{x_n - x_i}{x_n - x_0} f^{(k)}(x_i) + \frac{x_i - x_0}{x_n - x_0} f^{(k)}(x_i) = f^{(k)}(x_i), \quad k = 0, 1, \ldots, r_i.$$

□

In order to look at an example, denote p by

$$p_{\underbrace{0,\ldots,0}_{r_0+1},\underbrace{1,\ldots,1}_{r_1+1},\ldots,\underbrace{n,\ldots,n}_{r_n+1}},$$

and consider cubic interpolation with $x_0 = 0$, $x_1 = 1$, and $r_0 = r_1 = 1$. The iteration gives

$$\begin{aligned} p_{0011} &= (1-x)p_{001} + xp_{011}, \\ p_{001} &= (1-x)p_{00} + xp_{01}, \\ p_{011} &= (1-x)p_{01} + xp_{11}, \\ p_{01} &= (1-x)p_0 + xp_1, \end{aligned}$$

and the Taylor polynomial (3) gives

$$p_0 = f(0), \quad p_{00} = f(0) + xf'(0),$$

$$p_1 = f(1), \quad p_{11} = f(1) + (x-1)f'(1).$$

Therefore,

$$\begin{aligned} p(x) = (1-x)^2&(f(0) + xf'(0)) \\ &+ 2x(1-x)((1-x)f(0) + xf(1)) \\ &+ x^2(f(1) + (x-1)f'(1)). \end{aligned}$$

3 The Newton form

A Hermite interpolant can also be represented in Newton form, the advantage being that the divided differences can be computed just once, and the evaluation of the interpolant for any given x is relatively fast, requiring only $O(n)$ flops.

Defining the polynomial,

$$\omega_N(x) := (x-x_0)^{r_0+1} \cdots (x-x_{i-1})^{r_{i-1}+1}(x-x_i)^{r_n},$$

we can express the interpolant $p = p_N$ as

$$p_N(x) = p_{N-1}(x) + c_N\omega_N(x),$$

with c_N the leading coefficient of p_N. Continuing the recursion, we obtain the Newton form of the Hermite interpolant,

$$p_N(x) = \sum_{i=0}^{N} c_i \omega_i(x). \tag{5}$$

In the special case that $n = 0$, the leading coefficient of p_N is

$$c_N = f^{(r_0)}(x_0)/r_0!, \tag{6}$$

while if $n \geq 1$, the iterative interpolation algorithm, Theorem 2, gives a recursion for c_N, because, with $\mathrm{lc}(p)$ denoting the leading coefficient of p, equation (4) implies

$$\mathrm{lc}(p) = \frac{\mathrm{lc}(r) - \mathrm{lc}(q)}{x_n - x_0}. \tag{7}$$

In this way we can compute all the divided differences c_k required in (5), and due to (6) and (7) we now see that c_N is the divided difference

$$c_N = [\underbrace{x_0, \ldots, x_0}_{r_0+1}, \underbrace{x_1, \ldots, x_1}_{r_1+1}, \ldots, \underbrace{x_n, \ldots, x_n}_{r_n}]f.$$

Consider again the example of cubic interpolation, with $r_0 = r_1 = 1$. The Newton form of the interpolant is

$$\begin{aligned} p(x) = {} & [x_0]f + [x_0, x_0]f\,(x - x_0) + [x_0, x_0, x_1]f\,(x - x_0)^2 \\ & + [x_0, x_0, x_1, x_1]f\,(x - x_0)^2(x - x_1). \end{aligned}$$

These divided differences can be computed from

$$[x_0]f = f(x_0), \qquad [x_1]f = f(x_1),$$

$$[x_0, x_0]f = f'(x_0), \qquad [x_0, x_1]f = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \qquad [x_1, x_1]f = f'(x_1),$$

$$[x_0, x_0, x_1]f = \frac{[x_0, x_1]f - f'(x_0)}{x_1 - x_0}, \qquad [x_0, x_1, x_1]f = \frac{f'(x_1) - [x_0, x_1]f}{x_1 - x_0},$$

and

$$[x_0, x_0, x_1, x_1]f = \frac{[x_0, x_1, x_1]f - [x_0, x_0, x_1]f}{x_1 - x_0}.$$

From now on we can simplify notation and consider a sequence of points $x_0, \ldots, x_n$ in $[a, b]$, that are *distinct or not*. For each i, we let ρ_i be the *left-multiplicity* of x_i,

$$\rho_i = |\{0 \le j < i : x_j = x_i\}|,$$

i.e., the number of points in the sequence $x_0, \ldots, x_{i-1}$ that are equal to x_i. The Hermite interpolant to f is then the unique polynomial $p_n \in \pi_n$ such that

$$p_n^{(\rho_i)}(x_i) = f^{(\rho_i)}(x_i), \qquad i = 0, \ldots, n.$$

We have shown that

$$p_n(x) = p_{n-1}(x) + c_n \omega_n(x),$$

where

$$\omega_n(x) = (x - x_0) \cdots (x - x_{n-1}),$$

and c_n is the divided difference of f,

$$c_n = [x_0, x_1, \ldots, x_n]f.$$

This divided difference is symmetric in the points $x_0, \ldots, x_n$, and so we may assume that $x_i \le x_{i+1}$, in which case

$$[x_0, x_1, \ldots, x_n]f = \frac{[x_1, \ldots, x_n]f - [x_0, \ldots, x_{n-1}]f}{x_n - x_0}, \qquad \text{if } x_0 < x_n,$$

and

$$[x_0, x_1, \ldots, x_n]f = f^{(n)}(x_0)/n!, \qquad \text{if } x_0 = \cdots = x_n.$$

The Newton form of p is

$$p(x) = \sum_{i=0}^{n} [x_0, \ldots, x_i]f \, \omega_i(x),$$

and its error is

$$f(x) - p(x) = [x_0, \ldots, x_n, x]f \, (x - x_0) \cdots (x - x_n),$$

and there is some ξ in the smallest interval containing $x_0, \ldots, x_n$ and x such that

$$[x_0, \ldots, x_n, x]f = \frac{f^{(n+1)}(\xi)}{(n+1)!}.$$

As an example, we find the error of the cubic Hermite interpolant p_3 we studied previsouly, at the points x_0, x_1, with $x_0 < x_1$. For $x \in [x_0, x_1]$, if $f \in C^4[x_0, x_1]$, there is some $\xi \in [x_0, x_1]$ such that

$$e(x) := f(x) - p_3(x) = (x - x_0)^2 (x - x_1)^2 \frac{f^{(4)}(\xi)}{4!}.$$

Since

$$\max_{x_0 \le x \le x_1} (x - x_0)^2 (x - x_1)^2 = \frac{h^4}{16},$$

where $h = x_1 - x_0$, we deduce that

$$\max_{x_0 \le x \le x_1} |e(x)| \le \frac{h^4 M}{384},$$

where

$$M = \max_{x_0 \le y \le x_1} |f^{(4)}(y)|.$$

As a final remark, we note that by the common Newton form of both Hermite and Lagrange interpolation, we see that a Hermite interpolant is the limit of Lagrange interpolants as some points coallesce, provided f has sufficiently many derivatives.