Multivariate interpolation

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These notes study bivariate polynomial interpolation, focusing on tensor-product and lower set interpolation.

1 Tensor-product interpolation

Suppose that a real, bivariate function f is defined at the points $(x_i, y_j) \in \mathbb{R}^2$, with $x_0, \ldots, x_m \in \mathbb{R}$ distinct and $y_0, \ldots, y_n \in \mathbb{R}$ distinct. The set of all these points, $X = \{(x_i, y_j)\}$, forms a Cartesian, rectangular grid in the plane. Let $\pi_{m,n}$ denoting the linear space of all bivariate polynomials of the form

$$p(x, y) = \sum_{k=0}^{m} \sum_{l=0}^{n} c_{kl} x^k y^l. \tag{1}$$

We will show

Theorem 1 *There exists a unique p in $\pi_{m,n}$ that interpolates f on X, i.e., such that*

$$p(x_i, y_j) = f(x_i, y_j), \qquad i = 0, \ldots, m, \quad j = 0, \ldots, n. \tag{2}$$

Proof. Let I be the set of multi-indices, i.e., index pairs,

$$I = \{(k, l) : 0 \le k \le m, 0 \le l \le n\}.$$

Then, similar to the univariate case, we can express the $(m + 1)(n + 1)$ equations (2) in matrix form as

$$M\mathbf{c} = \mathbf{f},$$

where

$$M = [x_i^k y_j^l]_{(i,j),(k,l)\in I}, \qquad \mathbf{c} = [c_{kl}]_{(k,l)\in I}, \qquad \mathbf{f} = [f(x_i, y_j)]_{(i,j)\in I},$$

with respect to some ordering of the multi-indices in I, and $\mathbf{c}$ and $\mathbf{f}$ column vectors. The existence and uniqueness of p is then equivalent to the non-singularity of M. So, suppose $M\mathbf{c} = 0$ for some $\mathbf{c}$. Then the polynomial

$$q(x, y) = \sum_{k=0}^{m} \sum_{l=0}^{n} c_{kl} x^k y^l$$

is zero at every point (x_i, y_j). For each $j = 0, \dots, n$, let $p_j(x) = q(x, y_j)$. Then p_j belong to π_m and has zeros $x_0, \dots, x_m$. So by the Fundamental Theorem of Algebra, $p_j = 0$ and therefore its coefficients are zero, i.e.,

$$\sum_{l=0}^{n} c_{kl} y_j^l = 0, \qquad k = 0, \dots, m. \tag{3}$$

Then, for each $k = 0, \dots, m$, let

$$r_k(y) = \sum_{l=0}^{n} c_{kl} y^l.$$

Since $r_k \in \pi_n$ and has zeros $y_0, \dots, y_n$, the F.T.A. can again be applied to show that $r_k = 0$, and hence its coefficients are zero, i.e., $c_{kl} = 0$, $l = 0, \dots, n$. Hence $\mathbf{c} = 0$ and M is non-singular. □

The monomial form (1) is not the only way of representing p. If $\phi_0, \dots, \phi_m$ is any basis for π_m and $\psi_0, \dots, \psi_n$ any basis for π_n, we can form the (tensor-) products

$$B_{kl}(x, y) = \phi_k(x)\psi_l(y),$$

and represent p as

$$p(x, y) = \sum_{k=0}^{m} \sum_{l=0}^{n} c_{kl} B_{kl}(x, y).$$

For example, choosing the univariate Lagrange bases

$$\phi_k(x) = \prod_{\substack{r=0 \\ r\neq k}}^{m} \frac{x - x_r}{x_k - x_r}, \qquad \psi_l(y) = \prod_{\substack{s=0 \\ s\neq l}}^{n} \frac{y - y_s}{y_l - y_s},$$

means that

$$B_{kl}(x_i, y_j) = \phi_k(x_i)\psi_l(y_j) = \delta_{ki}\delta_{lj},$$

from which it follows that

$$p(x, y) = \sum_{k=0}^{m} \sum_{l=0}^{n} f(x_k, y_l) B_{kl}(x, y),$$

is the interpolant in $\pi_{m,n}$ to f.

2 Newton form

There is also a Newton representation for the interpolant, which at the same time leads to error formulas. Here, a divided difference of a function of several variables is formed by keeping one variable fixed and taking the indicated differences with respect to the free variable. Thus, for example,

$$[x_0, \ldots, x_m; y_0, \ldots, y_n]f = \frac{[x_1, \ldots, x_m; y_0, \ldots, y_n]f - [x_0, \ldots, x_{m-1}; y_0, \ldots, y_n]f}{x_m - x_0}$$

if $m \geq 1$, and

$$[x_0, \ldots, x_m; y_0, \ldots, y_n]f = \frac{[x_0, \ldots, x_m; y_1, \ldots, y_n]f - [x_0, \ldots, x_m; y_0, \ldots, y_{n-1}]f}{y_n - y_0}$$

if $n \geq 1$. If both $m \geq 1$ and $n \geq 1$ either formula can be used. In the case $m = n = 0$, $[x_0, y_0]f = f(x_0, y_0)$.

By the obvious extension of the Genocchi-Hermite formula to bivariate divided differences, there is some point (ξ, η) in the smallest rectangle D containing the points (x_k, y_l), $(k, l) \in I$ such that

$$[x_0, \ldots, x_k; y_0, \ldots, y_l]f = \frac{1}{k!l!} \frac{\partial^{k+l} f(\xi, \eta)}{\partial x^k \partial y^l},$$

provided the mixed derivative $\partial^{k+l} f / (\partial x^k \partial y^l)$ is continuous in D.

Using these bivariate divided differences, the polynomial interpolant and its error can be derived simultaneously from the corresponding formulas from the univariate case. We first expand f with respect to the x variable and the points $x_0, \ldots, x_m$:

$$f(x,y) = \sum_{k=0}^{m} \mu_k(x)[x_0, \ldots, x_k; y]f + R_1(x,y), \tag{4}$$

with remainder

$$R_1(x,y) = \mu_{m+1}(x)[x_0, \ldots, x_m, x; y]f.$$

Here,

$$\mu_0(x) := 1, \qquad \text{and} \qquad \mu_k(x) := (x - x_0)\cdots(x - x_{k-1}), \quad k \geq 1.$$

Then, treating $[x_0, \ldots, x_k; y]f$ as a function of y, we make a Newton expansion of it with respect to the y variable and the points $y_0, \ldots, y_n$:

$$[x_0, \ldots, x_k; y]f = \sum_{l=0}^{n} \nu_l(y)[x_0, \ldots, x_k; y_0, \ldots, y_l]f + R_{2,k}(y),$$

with

$$R_{2,k}(y) = \nu_{n+1}(y)[x_0, \ldots, x_k; y_0, \ldots, y_n, y]f,$$

where

$$\nu_0(y) := 1, \qquad \text{and} \qquad \nu_l(y) := (y - y_0)\cdots(y - y_{l-1}), \quad l \geq 1.$$

Substituting this into (4) gives

$$f(x,y) = p(x,y) + R(x,y),$$

where

$$p(x,y) = \sum_{k=0}^{m} \sum_{l=0}^{n} \mu_k(x)\nu_l(y)[x_0, \ldots, x_k; y_0, \ldots, y_l]f, \tag{5}$$

and

$$R(x,y) = R_1(x,y) + R_2(x,y), \tag{6}$$

with

$$R_2(x,y) = \nu_{n+1}(y) \sum_{k=0}^{m} \mu_k(x)[x_0, \ldots, x_k; y_0, \ldots, y_n, y]f.$$

Since

$$R_1(x_i, y) = 0, \qquad i = 0, \ldots, m,$$

and

$$R_2(x, y_j) = 0, \qquad j = 0, \ldots, n,$$

it follows that $R(x_i, y_j) = 0$ at the $(m+1)(n+1)$ points (x_i, y_j), $(i, j) \in I$. Therefore p in (8) is the unique interpolant to f in $\pi_{m,n}$, and R its error. To simplify the expression for the latter, a further Newton expansion shows that

$$\sum_{k=0}^{m} \mu_k(x)[x_0, \ldots, x_k; y_0, \ldots, y_n, y]f =$$
$$[x; y_0, \ldots, y_n, y]f - \mu_{m+1}(x)[x_0, \ldots, x_m, x; y_0, \ldots, y_n, y]f,$$

and substituting this into the second term in (9) gives

Theorem 2 *The interpolant p can be expressed in the Newton form (8), and the error, $R = f - p$, is*

$$\begin{aligned} R(x, y) = \mu_{m+1}(x)[x_0, \ldots, x_m, x; y]f + \nu_{n+1}(y)[x; y_0, \ldots, y_n, y]f \\ - \mu_{m+1}(x)\nu_{n+1}(y)[x_0, \ldots, x_m, x; y_0, \ldots, y_n, y]f. \end{aligned}$$

A corollary of this three-term error formula is that

$$\begin{aligned} R(x, y) = \frac{\mu_{m+1}(x)}{(m+1)!} \frac{\partial^{m+1} f(\xi, y)}{\partial x^{m+1}} + \frac{\nu_{n+1}(y)}{(n+1)!} \frac{\partial^{n+1} f(x, \eta)}{\partial y^{n+1}} \\ - \frac{\mu_{m+1}(x)\nu_{n+1}(y)}{(m+1)!(n+1)!} \frac{\partial^{m+n+2} f(\xi', \eta')}{\partial x^{m+1} \partial y^{n+1}}. \end{aligned}$$

3 Lower set interpolation

We now consider interpolation on a more general set of points, a subset of a rectangular grid. For any non-increasing sequence,

$$n_0 \geq n_1 \geq \cdots \geq n_m \geq 0, \tag{7}$$

let $L \subset \mathbb{N}_0^2$ be the set of multi-indices,

$$L = \{(i, j) : 0 \leq i \leq m, 0 \leq j \leq n_i\}.$$

Such a set L is called a *lower set*, and it is closed under the usual partial ordering of multi-indices. Two multi-indices (i,j) and (k,l) in $\mathbb{N}_0^2$ are ordered, with $(i,j) \le (k,l)$, if both $i \le k$ and $j \le l$. We see then that L is closed in the sense that if $(k,l) \in L$ and $(i,j) \le (k,l)$ then $(i,j) \in L$.

We want to show that we can uniquely interpolate a function f on the points

$$X_L := \{(x_i, y_j) : (i,j) \in L\},$$

from the linear space π_L of polynomials of the form

$$p(x,y) = \sum_{k=0}^{m} \sum_{l=0}^{n_k} c_{kl} x^k y^l = \sum_{(k,l)\in L} c_{kl} x^k y^l.$$

To show this, we return to the first Newton expansion (4) and now expand the term $[x_0, \ldots, x_k; y]f$ as a Newton polynomial of degree n_k:

$$[x_0, \ldots, x_k; y]f = \sum_{l=0}^{n_k} \nu_l(y)[x_0, \ldots, x_k; y_0, \ldots, y_l]f + R_{2,k}(y),$$

with

$$R_{2,k}(y) = \nu_{n_k+1}(y)[x_0, \ldots, x_k; y_0, \ldots, y_{n_k}, y]f.$$

Substituting this into (4) gives

$$f(x,y) = p(x,y) + R(x,y),$$

where

$$p(x,y) = \sum_{k=0}^{m} \sum_{l=0}^{n_k} \mu_k(x)\nu_l(y)[x_0, \ldots, x_k; y_0, \ldots, y_l]f, \tag{8}$$

and

$$R(x,y) = R_1(x,y) + R_2(x,y), \tag{9}$$

with

$$R_2(x,y) = \sum_{k=0}^{m} \mu_k(x)\nu_{n_k+1}(y)[x_0, \ldots, x_k; y_0, \ldots, y_{n_k}, y]f.$$

Clearly, $p \in \pi_L$ and to show that p interpolates f on X_L we show that $R(x_i, y_j) = 0$ for all $(i,j) \in L$. So let $(i,j) \in L$. As in the tensor-product case,

$R_1(x_i, y_j) = 0$ since $\mu_{m+1}(x_i) = 0$. It remains to show that $R_2(x_i, y_j) = 0$. Since $\mu_k(x_i) = 0$ if $k > i$, we see that

$$R_2(x_i, y_j) = \sum_{k=0}^{i} \mu_k(x_i)\nu_{n_k+1}(y_j)[x_0, \dots, x_k; y_0, \dots, y_{n_k}, y_j]f.$$

Thus $k \leq i$ in the sum, and so, due to condition (7), $n_k \geq n_i$. Therefore,

$$j \leq n_i \leq n_k,$$

which implies that $\nu_{n_k+1}(y_j) = 0$. Thus $R_2(x_i, y_j) = 0$ as claimed.

The error can be expressed in terms of derivatives as

$$\begin{aligned} R(x, y) = {} & \frac{\mu_{m+1}(x)}{(m+1)!} \frac{\partial^{m+1} f(\xi, y)}{\partial x^{m+1}} \\ & + \sum_{k=0}^{m} \frac{\mu_k(x)\nu_{n_k+1}(y)}{k!(n_k+1)!} \frac{\partial^{k+n_k+1} f(\xi', \eta')}{\partial x^k \partial y^{n_k+1}}. \end{aligned}$$