Newton interpolation

Michael S. Floater

January 27, 2014

These notes derive the Newton form of polynomial interpolation, and study the associated divided differences.

1 The Newton form

Recall that for distinct points $x_0, x_1, \ldots, x_n$, and a real function f defined at these points, there is a unique polynomial interpolant $p_n \in \pi_n$. The idea of Newton interpolation is to build up p_n from the interpolant p_{n-1} for $n \geq 1$. Defining the polynomial,

$$\omega_n(x) := (x - x_0)(x - x_1) \cdots (x - x_{n-1}),$$

we can express the interpolant p_n as

$$p_n(x) = p_{n-1}(x) + c_n \omega_n(x), \tag{1}$$

for some constant $c_n \in \mathbb{R}$. To see this, let $x = x_i$ for some $i \in \{0, 1, \ldots, n-1\}$. Since $\omega_n(x_i) = 0$,

$$p_n(x_i) = p_{n-1}(x_i) = f(x_i).$$

On the other hand, since $\omega_n(x_n) \neq 0$, we can determine c_n to solve the remaining interpolation condition, $p_n(x_n) = f(x_n)$. This condition becomes

$$f(x_n) = p_{n-1}(x_n) + c_n \omega_n(x_n),$$

and the solution is to take

$$c_n = \frac{f(x_n) - p_{n-1}(x_n)}{\omega_n(x_n)}.$$

We can continue in this way, expressing p_{n-1} in terms of p_{n-2}, and so on. Since clearly $p_0(x) = f(x_0)$, we deduce that

$$p_n(x) = \sum_{k=0}^{n} c_k \omega_k(x), \tag{2}$$

where $\omega_0(x) := 1$ and $c_0 = f(x_0)$. This is the so-called *Newton form* of the interpolant. Once we have found the coefficients c_k, we can adapt Horner's rule to evaluate p_n. For example, we compute p_3 as

$$p_3(x) = c_0 + (x - x_0)(c_1 + (x - x_1)(c_2 + (x - x_2)c_3)).$$

2 Divided differences

Consider the coefficient c_n in equation (1). Since

$$\omega_n(x) = x^n + \text{lower order terms},$$

ω_n has leading coefficient 1. Since also $p_{n-1} \in \pi_{n-1}$, it follows from equation (1) that c_n is the *leading coefficient* of p_n, and, more generally, c_k is the leading coefficient of the polynomial in π_k that interpolates f at $x_0, x_1, \ldots, x_k$. To indicate this dependency we express c_k as

$$c_k = [x_0, x_1, \ldots, x_k]f,$$

and we use the fact that it is the leading coefficient of p_k to find a convenient way of computing it, for $k \geq 1$. Recall the iterative interpolation of the previous lecture. If $q_{k-1} \in \pi_{k-1}$ is the interpolant to f on the points $x_1, x_2, \ldots, x_k$, we can express the interpolant p_k as

$$p_k(x) = \frac{x_k - x}{x_k - x_0} p_{k-1}(x) + \frac{x - x_0}{x_k - x_0} q_{k-1}(x).$$

Since p_{k-1} has leading coefficient $[x_0, x_1, \ldots, x_{k-1}]f$ and q_{k-1} has leading coefficient $[x_1, x_2, \ldots, x_k]f$, equating the leading coefficients of the two sides of the equation gives

$$[x_0, x_1, \ldots, x_k]f = \frac{[x_1, x_2, \ldots, x_k]f - [x_0, x_1, \ldots, x_{k-1}]f}{x_k - x_0}. \tag{3}$$

For this reason, the coefficient c_k is known as the *divided difference* of f at the points $x_0, x_1, \dots, x_k$. This formula can be used to compute the *divided difference table*, which provides the divided differences required in the Newton form (2). The table for $n = 3$ is shown below.

$$
\begin{array}{llll}
[x_0]f & & & \\
[x_1]f & [x_0, x_1]f & & \\
[x_2]f & [x_1, x_2]f & [x_0, x_1, x_2]f & \\
[x_3]f & [x_2, x_3]f & [x_1, x_2, x_3]f & [x_0, x_1, x_2, x_3]f
\end{array}
$$

Each entry in the table is computed from two entries in the previous column: the one in the row above and the one in the same row. Hence the complete table can be constructed, for example, row by row, or column by column. The divided differences required in (2) are on the top diagonal. The first examples are

$$
\begin{aligned}
[x_0]f &= f(x_0), \\
[x_0, x_1]f &= \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \\
[x_0, x_1, x_2]f &= \left(\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0} \right) \Big/ (x_2 - x_0).
\end{aligned}
$$

If we add a new point x_{n+1} to the interpolation, we can compute $p_{n+1}(x)$ from $p_n(x)$ using the fact that

$$p_{n+1}(x) = p_n(x) + [x_0, \dots, x_{n+1}]f\omega_{n+1}(x).$$

We can compute $\omega_{n+1}(x)$ as

$$\omega_{n+1}(x) = (x - x_n)\omega_n(x),$$

and find $[x_0, \dots, x_{n+1}]f$ by computing one more row of the table.

Note also that since the interpolant p_k is independent of the ordering of the point $x_0, x_1, \dots, x_k$, the divided difference $[x_0, \dots, x_k]f$ is *symmetric* in its arguments: it is unchanged if x_i and x_j are swapped for any $i \neq j$. This fact can be used to derive alternative recursion formulas. For example, by swapping x_0 and x_{k-1} in (3) we obtain

$$[x_0, x_1, \dots, x_k]f = \frac{[x_0, x_1, \dots, x_{k-2}, x_k]f - [x_0, x_1, \dots, x_{k-1}]f}{x_k - x_{k-1}}. \qquad (4)$$

The symmetry property means, more generally, that a divided difference of order k can be expressed as the difference between the two divided differences of order $k-1$ obtained by removing each of any two of the nodes, and dividing by the difference between the nodes.

We can also obtain an explicit formula for divided differences from the Lagrange form of interpolation. Recall the Lagrange form of p_n from the previous lecture,

$$p_n(x) = \sum_{i=0}^{n} \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} f(x_i).$$

By extracting the leading coefficient from the right hand side, we deduce that

$$[x_0, \ldots, x_n]f = \sum_{i=0}^{n} \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{1}{x_i - x_j} f(x_i). \tag{5}$$

3 Equidistant points

A frequently occurring case of divided differences is when the points are equidistant. If $x_i = x_0 + ih$, $i = 1, \ldots, n$, for some $h > 0$, we find

$$[x_0, x_1, \ldots, x_n]f = \frac{\Delta^n f_0}{h^n n!}, \tag{6}$$

where $f_i := f(x_i)$ and Δ is the forward difference operator, defined by

$$\Delta f_0 := f_1 - f_0,$$

and for $k > 1$,

$$\Delta^k f := \Delta(\Delta^{k-1} f_0).$$

This means that

$$\Delta^2 f_0 = f_2 - 2f_1 + f_0, \quad \Delta^3 f_0 = f_3 - 3f_2 + 3f_1 - f_0,$$

and more generally,

$$\Delta^n f_0 = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} f_i.$$

Equation (6) is easily established by induction on n from the recursion formula (3).

4 Interpolation error

An advantage of the Newton form of interpolation is that it also provides a formula for the interpolation error, $f(x) - p_n(x)$.

Theorem 1 *For x distinct from $x_0, \ldots, x_n$,*

$$f(x) - p_n(x) = [x_0, \ldots, x_n, x]f\omega_{n+1}(x).$$

Proof. Let $p_{n+1} \in \pi_{n+1}$ be the interpolant to f on the points $x_0, x_1, \ldots, x_n, x$. Then

$$p_{n+1}(y) = p_n(y) + c_{n+1}\omega_{n+1}(y), \qquad y \in \mathbb{R},$$

with c_{n+1} the leading coefficient of p_{n+1}, i.e.,

$$c_{n+1} = [x_0, \ldots, x_n, x]f.$$

Letting $y = x$ and using the fact that $p_{n+1}(x) = f(x)$ gives the result. $\square$

5 Genocchi-Hermite formula

We have obtained a formula for the error of interpolation in terms of a divided difference of f. If f is sufficiently smooth, the error can also be expressed in terms of derivatives of f. One way of doing this is to use the Genocchi-Hermite formula.

Theorem 2 *For $n \geq 1$, let $x_0, \ldots, x_n$ be distinct points and suppose $f^{(n)}$ is continuous in an interval containing them. Then*

$$[x_0, \ldots, x_n]f = \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{n-1}} f^{(n)}(\xi) dt_n \cdots dt_2 dt_1, \tag{7}$$

where

$$\xi = x_0 + \sum_{i=1}^{n} t_i(x_i - x_{i-1}). \tag{8}$$

Proof. We prove the formula by induction on n. For $n = 1$ we use the integral representation,

$$[x_0, x_1]f = \frac{1}{x_1 - x_0} \int_{x_0}^{x_1} f'(x)\, dx,$$

and then make the change of variable,

$$x = x_0 + t(x_1 - x_0),$$

giving

$$[x_0, x_1]f = \int_0^1 f'(x_0 + t(x_1 - x_0))\, dt.$$

For $n \geq 2$,

$$[x_0, \ldots, x_n]f = \frac{[x_0, \ldots, x_{n-2}, x_n]f - [x_0, \ldots, x_{n-2}, x_{n-1}]f}{x_n - x_{n-1}},$$

and so, by the induction hypothesis,

$$[x_0, \ldots, x_n]f = \int_0^1 \cdots \int_0^{t_{n-2}} \frac{f^{(n-1)}(\xi_1) - f^{(n-1)}(\xi_0)}{x_n - x_{n-1}}\, dt_{n-1} \cdots dt_1, \qquad (9)$$

where

$$\xi_0 = x_0 + \sum_{i=1}^{n-1} t_i(x_i - x_{i-1}),$$
$$\xi_1 = x_0 + \sum_{i=1}^{n-2} t_i(x_i - x_{i-1}) + t_{n-1}(x_n - x_{n-2}).$$

Using the fact that

$$f^{(n-1)}(\xi_1) - f^{(n-1)}(\xi_0) = \int_{\xi_0}^{\xi_1} f^{(n)}(\xi)\, d\xi,$$

and changing the variable ξ to t_n via (8), gives

$$\frac{f^{(n-1)}(\xi_1) - f^{(n-1)}(\xi_0)}{x_n - x_{n-1}} = \int_0^{t_{n-1}} f^{(n)}(\xi)\, dt_n.$$

Substituting this into (9) gives the result. $\square$

One can view the integral in the theorem as an integral over the simplex $S \subset \mathbb{R}^n$ with vertices

$$v_i = (\underbrace{1, \ldots, 1}_{i}, \underbrace{0, \ldots, 0}_{n-i}), \qquad i = 0, 1, \ldots, n.$$

The theorem then says that

$$[x_0, \ldots, x_n]f = \int_{t \in S} f^{(n)}(\xi(t))\, dt,$$

where $\xi : \mathbb{R}^n \to \mathbb{R}$ is the linear polynomial such that $\xi(v_i) = x_i$, $i = 0, 1, \ldots, n$.

Corollary 1 *If $f^{(n)}$ is continuous in the smallest interval $[a, b]$ containing $x_0, \ldots, x_n$, there is some $\xi \in (a, b)$ such that*

$$[x_0, \ldots, x_n]f = \frac{f^{(n)}(\xi)}{n!}.$$

Proof. By the mean value theorem for integrals, there is some $\xi \in (a, b)$ such that

$$[x_0, \ldots, x_n]f = f^{(n)}(\xi) \int_{t \in S} 1\, dt,$$

and the integral on the right is the volume of S, which is $1/n!$. □

6 Interpolation error

We now return to the error in polynomial interpolation and consider again the error formula in Theorem 1. Due to Corollary 1, we deduce the following.

Corollary 2 *If $f^{(n)}$ is continuous in an interval (a, b) containing the distinct points $x_0, \ldots, x_n, x$, there is some $\xi \in (a, b)$ such that*

$$f(x) - p_n(x) = (x - x_0) \cdots (x - x_n) \frac{f^{(n+1)}(\xi)}{(n+1)!}.$$

As an example, consider the error in linear interpolation on the points x_0, x_1, with $x_0 < x_1$, when $f \in C^2[x_0, x_1]$. For $x \in [x_0, x_1]$ there is some $\xi \in [x_0, x_1]$ such that

$$e(x) := f(x) - p_1(x) = (x - x_0)(x - x_1) \frac{f''(\xi)}{2!}.$$

Since $|(x - x_0)(x - x_1)|$ attains its maximum value at $x = (x_0 + x_1)/2$,

$$\max_{x_0 \le x \le x_1} |(x - x_0)(x - x_1)| = \frac{h^2}{4},$$

where $h = x_1 - x_0$, and it follows that

$$\max_{x_0 \le x \le x_1} |e(x)| \le \frac{h^2 M}{8},$$

where

$$M = \max_{x_0 \le y \le x_1} |f''(y)|.$$

7 Non-distinct points

Another consequence of the Genocchi-Hermite formula is that it shows that $[x_0, \ldots, x_n]f$ is a continuous function of the points $x_0, \ldots, x_n$ in any interval in which $f^{(n)}$ is continuous. Thus, the formula defines the unique continuous extension of $[x_0, \ldots, x_n]f$ to *non-distinct* points $x_0, \ldots, x_n$ when $f^{(n)}$ is continuous. A special case is

$$[\underbrace{x, x, \ldots, x}_{n+1}]f = \frac{f^{(n)}(x)}{n!}.$$

If not all the points are equal, we can apply recursion: if $f^{(n-1)}$ is continuous in $[a, b]$, and $x_0, x_1, \ldots, x_n$ are any points in $[a, b]$, not-necessarily distinct, but x_0 and x_n are distinct, then

$$[x_0, \ldots, x_n]f = \frac{[x_1, \ldots, x_n]f - [x_0, \ldots, x_{n-1}]f}{x_n - x_0}.$$