1) $\begin{vmatrix} 1 & 2 & -1 \\ \beta & -1 & 2 \\ 2 & 1 & \beta \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ 1 & \beta \end{vmatrix} - \beta \begin{vmatrix} 2 & -1 \\ 1 & \beta \end{vmatrix} + 2 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$

$= -\beta - 2 - \beta(2\beta + 1) + 2(4-1)$

$= -2\beta^2 - 2\beta + 4 = 0 \quad \Rightarrow \beta = \dfrac{2 \pm \sqrt{4+32}}{-4}$

$= \dfrac{2 \pm 6}{-4} = \begin{cases} -2 \\ 1 \end{cases}$

For $\beta = -2$:

$\left.\begin{aligned} x + 2y - z &= -2 \\ -2x - y + 2z &= 0 \\ 2x + y - 2z &= 1 \end{aligned}\right\}$ opplagt ingen løsning

For $\beta = 1$:

$\left.\begin{aligned} x + 2y - z &= 1 \\ x - y + 2z &= 0 \\ 2x + y + z &= 1 \end{aligned}\right.$ $\begin{aligned} 3x + 3y &= 2 \\ -3x - 3y &= -2 \end{aligned}$ gir $x + y = \frac{2}{3}$

dvs $y = \frac{2}{3} - x$

Innsatt: $z = x + 2y - 1 = x + \frac{4}{3} - 2x - 1 = -x + \frac{1}{3}$

dvs $(x,y,z) = (x, \frac{2}{3} - x, -x + \frac{1}{3}) = \underline{(0, \frac{2}{3}, \frac{1}{3}) + x(1,-1,-1)}$

2) Kar. pol: $\lambda^2 + \lambda - 2 = 0$ gir $\lambda = \dfrac{-1 \pm \sqrt{1+8}}{2} = \dfrac{-1 \pm 3}{2} = \begin{cases} 1 \\ -2 \end{cases}$

$\Rightarrow \quad x_n^0 = C \cdot 1^n + D(-2)^n = C + D(-2)^n$

$x_n^s = An$: $A(n+2) + A(n+1) - 2An = 2A + A = 3A = 3$ gir $A = 1$

$\Rightarrow \quad x_n = C + D(-2)^n + n$

$\left.\begin{aligned} 2 &= x_0 = C + D \\ 0 &= x_1 = C - 2D + 1 \end{aligned}\right\}$ $C = D = 1$

$$\underline{\underline{x_n = 1 + (-2)^n + n}}$$

3)

$$v' + kv = g \quad \Rightarrow \quad v = e^{-kt}\int e^{kt} \cdot g \, dt$$

$$= e^{-kt}\left(\frac{g}{k}e^{kt} + C\right)$$

$$= \frac{g}{k} + Ce^{-kt}$$

$$0 = v(0) = \frac{g}{k} + C \quad \Rightarrow \quad C = -\frac{g}{k}$$

$$\underline{v(t) = \frac{g}{k}\left(1 - e^{-kt}\right)}$$

Når $t \to \infty$, går $v(t) \to \underline{\underline{\frac{g}{k}}}$

Alternativt, bruker vi likningen, stabilisering betyr $v' = 0$

$$\Rightarrow \quad kv = g \quad \text{eller} \quad \underline{\underline{v = \frac{g}{k}}}$$

4)

$$y' = \lambda x y^2$$

Separabel: $\displaystyle\int \frac{dy}{y^2} = \int \lambda x \, dx = \frac{1}{2}\lambda x^2 + C$

$$\parallel$$

$$-\frac{1}{y} \quad \Rightarrow \quad y = -\frac{1}{\frac{1}{2}\lambda x^2 + C}$$

$$1 = y(0) = -\frac{1}{0 + C} \quad \Rightarrow \quad C = -1$$

$$\Rightarrow \quad y = \frac{1}{1 - \frac{1}{2}\lambda x^2} = \underline{\underline{\frac{2}{2 - \lambda x^2}}}$$

Når $x \to \infty$ vil $y \to 0$ siden $\displaystyle\frac{2}{2-\lambda x^2} = \frac{\frac{2}{x^2}}{\frac{2}{x^2} - \lambda}$

5) Kar.pol $r^2 + \frac{1}{2}r - 5 = 0$ gir $r = \frac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} + 20}}{2}$

$= -\frac{1}{4} \pm \frac{9}{4} = \begin{cases} 2 \\ -\frac{5}{2} \end{cases}$

$\Rightarrow \quad y = Ce^{2x} + De^{-\frac{5}{2}x}$

$$\left.\begin{aligned} 3 &= y(0) = C + D \\ -3 &= y'(0) = 2C - \tfrac{5}{2}D \end{aligned}\right\} \quad C = 1,\ D = 2$$

$\Rightarrow \quad \underline{y = e^{2x} + 2e^{-\frac{5}{2}x}}$

6) Kar. pol $r^2 + 2r + 10 = 0$ gir $r = \frac{-2 \pm \sqrt{4 - 40}}{2}$

$= -1 \pm 3i$

$\Rightarrow \quad y = e^{-x}(C\cos(3x) + D\sin(3x))$

$$\left.\begin{aligned} 0 &= y(0) = C \\ 3 &= y'(0) = -e^0 D\sin(0) + e^0 \cdot 3D = 3D \end{aligned}\right\} \quad \cancel{C}\,D = 1,\ C = 0$$

$\Rightarrow \quad \underline{y = e^{-x}\sin(3x)}$

7) $y' + \frac{1}{x}y = \cos x$, $x > 0$ gir

$y = e^{-\int \frac{1}{x}dx} \int e^{\int \frac{1}{x}dx} \cos x \, dx = e^{-\ln x} \int e^{\ln x} \cos x \, dx$

$= \frac{1}{x} \int x\cos x \, dx = \frac{1}{x}\left(x\sin x - \int \sin x \, dx\right)$

$= \sin x + \frac{1}{x}(\cos x + C) = \sin x + \frac{\cos x}{x} + \frac{C}{x}$

$0 = y(\pi) = \sin\pi + \frac{\cos\pi}{\pi} + \frac{C}{\pi} = -\frac{1}{\pi} + \frac{C}{\pi} \Rightarrow C = 1$

$\Rightarrow \quad \underline{y = \sin x + \frac{\cos x}{x} + \frac{1}{x}}$

8) $\frac{d}{dt}\left(\frac{1}{2}z\right) = \lambda\sqrt{z}$, $\lambda < 0$, $t \geq 0$

Separabel , $\sqrt{z} = \int \frac{dz}{2\sqrt{z}} = \int \lambda dt = \lambda t + C$

$\Rightarrow \quad z = (\lambda t + C)^2$

$1 = z(0) = (0 + C)^2 = C^2 \quad \Rightarrow \quad C = \pm 1$.

For $C = -1$ får vi $z = (\lambda t + (-1))^2 = (\lambda t - 1)^2$, men $\sqrt{z} = -(\lambda t - 1)$ siden $\lambda t - 1 < 0$ og vi skal ha positiv rot.

$\Rightarrow \quad \frac{d}{dt}\left(\frac{1}{2}z\right) = \frac{1}{2} 2(\lambda t - 1) \cdot \lambda = \lambda(\lambda t - 1) = -\lambda\sqrt{z}$

Som ikke gir en løsning.

For $C = 1$ får vi $z = (\lambda t + 1)^2$ og $\frac{1}{2}\frac{d}{dt} z = \lambda(\lambda t + 1) = \lambda\sqrt{z}$

Siden $\lambda t + 1 \geq 0$ når $0 \leq t \leq -\frac{1}{\lambda}$, så $z(t) = (\lambda t + 1)^2$ er løsn.

$z(T) = (\lambda T + 1)^2 = 0$ gir $T = -\frac{1}{\lambda}$