

MAT1001, Fall 2011

Oblig 1

Deadline: Thursday, September 22, 1430 pm

You may work on these exercises together with your fellow students, but each of you has to hand in his/her own version of the solution to the assignment for approval. Please put your name and course code (MAT 1001) on your answer sheet. The assignments must be handed in to the Mathematics Department's reception by the given date and time. The time limit is absolute and assignments will not be accepted after the deadline unless an agreement has been made with the administration beforehand.

This exercise is about choices of refreshments in the breaks. How does your choice of coffee, tea or water in one break, affect your choice in the next break?

The dynamics of choices is as follows: Of the students who choose coffee in one break, 20% choose tea in the next break and 10% choose water. Of those who choose tea in one break, 10% choose coffee and 10% choose water. Of those who choose water in one break, 20% choose coffee in the next break and 30% choose tea (Notice that there are no empirical evidence for the given figures, they are only chosen as an example). The rest will just repeat their last choice. Let x denote the number of coffee drinkers, y the number of tea drinkers, and z the number of water drinkers. The indices give the number of the break, such that x_1 denotes the number of coffee drinkers in the first break, z_3 the number of water drinkers in the third break, etc. The distribution between the three choices in break n is given by a column vector:

$$P_n = \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$$

The dynamics of the choices is given by a transition matrix M

$$M = \begin{pmatrix} 0,7 & 0,1 & 0,2 \\ 0,2 & 0,8 & 0,3 \\ 0,1 & 0,1 & 0,5 \end{pmatrix}$$

i.e the choices for break $n + 1$ is computed on bases of the choices for break n by the matrix product

$$P_{n+1} = M \cdot P_n$$

- The matrix M is our model for the dynamics of the choices. Explain the entries in the matrix M and their connection to the percentages given in the introduction.
- Compute the product M^2 . (Why do we need this? Well, it describes what has happened to the distribution from one break to the second next. To understand what happens to the distribution in a longer run, we need even higher powers of M , but we shall attack that problem from a different point of view.)
- Find all eigenvalues $\lambda_1, \lambda_2, \lambda_3$ for M and their corresponding eigenvectors v_1, v_2, v_3 . (In our case the characteristic polynomial is of degree three. Even

if Cardano already in 1545 presented a general formula for the solution of polynomial equations of degree three, we simplify your problem by telling you that $\lambda = 1$ is an eigenvalue. The corresponding eigenvector is

$$v_1 = \begin{pmatrix} 7 \\ 13 \\ 4 \end{pmatrix}$$

Thus you know one of the solutions to the characteristic equation and by long division of polynomials (or some other method) you should be able to find the two remaining solutions. An example at the end of the sheet illustrates long division of polynomials.

- d) Compute the determinant $\det(M)$ of the matrix M by the general formula. A general result in linear algebra says that the determinant of a matrix equals the product of the eigenvalues. Check this result in our case. (This exercise is a test of your skills in the curriculum, and we will not need this computation for anything else in this assignment.)
- e) Next we study a more specific example. We consider 240 students and their choices of refreshments in the breaks. The distribution between coffee, tea and water in the first break is given by $x_1 = 180$ coffee, $y_1 = 20$ tea and $z_1 = 40$ water, i.e.

$$P_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 180 \\ 20 \\ 40 \end{pmatrix}$$

Compute P_2 and P_3 . (Notice that we allow non-integer persons, e.g. 109,6 persons)

- f) To see what happens to the distribution in the long run (many breaks later) we shall use what is called *spectral theory*, i.e. a clever way to exhibit information from the eigenvectors and -values of the matrix M . First we express the initial vector P_1 as a linear combination of eigenvectors, i.e. write

$$P_1 = a_1 v_1 + a_2 v_2 + a_3 v_3$$

as a linear combination of the eigenvectors you found in exercise c), for suitable values of a_1, a_2 and a_3 .

- g) We are going to use the decomposition of P_1 in exercise f) to compute $P_n = M^{n-1} P_1$ for all values of n : Find a formula for P_n using the entries of P_1 , the eigenvalues of M and n . When you have established the formula you can use it to describe what happens to the distribution in the long run, i.e. compute the limit of P_n when $n \rightarrow \infty$.
- h) This last exercise is more theoretical than the first seven exercises. The matrix M is special in the sense that the sum of each column equals 1. This implies in fact that 1 is an eigenvalue for M . Can you argue for this statement?

Each exercise a) - h) counts 0-5 points, all together a maximum of 40 points. Your Oblig 1 is approved if your score is 20 points or more.

An example to illustrate long division of polynomials.

Let us compute $(x^3 - 2x^2 + 4x + 7) : (x + 1)$. The result should be a polynomial of degree two. We start by the highest power of x , i.e. x^3 and estimates the "quotient" of $x^3 - 2x^2 + 4x + 7$ by $x + 1$, when restricting ourselves to the highest powers. The answer is x^2 . As for ordinary long division we multiply and get

$$\begin{array}{r} x^3 - 2x^2 + 4x + 7 : (x + 1) = x^2 \\ \underline{x^3 + x^2} \end{array}$$

Subtracting and pulling down the "next" term gives

$$\begin{array}{r} x^3 - 2x^2 + 4x + 7 : (x + 1) = x^2 \\ \underline{x^3 + x^2} \\ -3x^2 + 4x \end{array}$$

The "quotient" of $-3x^2 + 4x$ by $x + 1$ is $-3x$. Multiplication of $x + 1$ by $-3x$ gives $-3x^2 - 3x$, this we subtract from $-3x^2 + 4x$;

$$\begin{array}{r} x^3 - 2x^2 + 4x + 7 : (x + 1) = x^2 - 3x \\ \underline{x^3 + x^2} \\ -3x^2 + 4x \\ \underline{-3x^2 - 3x} \\ 7x + 7 \end{array}$$

Finally we see that $x + 1$ has to be multiplied by 7 for to get $7x + 7$. Thus

$$\begin{array}{r} x^3 - 2x^2 + 4x + 7 : (x + 1) = x^2 - 3x + 7 \\ \underline{x^3 + x^2} \\ -3x^2 + 4x \\ \underline{-3x^2 - 3x} \\ 7x + 7 \\ \underline{7x + 7} \\ 0 \end{array}$$