MAT1001, Fall 2012

Oblig 1

Deadline: Thursday, September 20, 1430 pm

You may work on these exercises together with your fellow students, but each of you has to hand in his/her own version of the solution to the assignment for approvement. Please put your name and course code (MAT 1001) on your answer sheet. The assignments must be handed in to the Mathematics Department's reception by the given date and time. The time limit is absolute and assignments will not be accepted after the deadline unless an agreement has been made with the administration beforehand.

Each exercise a) - h) counts 2-8 points, all together a maximum of 40 points. Your Oblig 1 is approved if your score is 20 points or more, and if you have tried to solve exercises a)-g).

This exercise is about the student's time of entering the auditorium for a lecture. The teacher will be present some minutes before the lecture to prepare the computer, the screen, etc. The students are divided into three groups. Those who come to the room before the teacher, those who come during the teacher's preparation, and those who are too late. The mathematical model describes the dynamic in the students choice of entering time from one lecture to the next.

The dynamic is as follows: The majority, i.e. 70%, of those who came early will be early also for the next lecture, 10% will be late, and 20% will be entering the room during the teacher's preparations. Of those who came during the preparations, 80% will enter during the preparations also for the next lecture, but 10% will be early and 10% will be late. For the late coming group the figures for the next lecture are 50% coming late, 30% entering during the preparations and 20% before the teacher.

Let x denote the number of early students, y the number of students entering the room during the teachers preparations, and z the number of late students. The indices give the number of the lecture, such that x_1 denotes the number of early entering students at the first lecture, z_3 the number of late entering students at the third lecture, etc. The distribution between the three possibilities for lecture n is given by a coloum vector:

$$P_n = \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$$

The dynamics of the choices is given by a transition matrix M

$$M = \begin{pmatrix} 0,7 & 0,1 & 0,2 \\ 0,2 & 0,8 & 0,3 \\ 0,1 & 0,1 & 0,5 \end{pmatrix}$$

i.e the distribution between the three categories for lecture $n+1$ is computed on bases of the distribution for lecture n by the matrix product

$$P_{n+1} = M \cdot P_n$$

a) (2 points) The matrix M is our model for the dynamics of the distribution. Explain the entries in the matrix M and their connection to the percentages given in the introduction.

b) (3 points) Compute the product M^2. (Why do we need this? Well, it describes what has happened to the distribution from one lecture to the next. To understand what happens to the distribution in a longer run, we need even higher powers of M, but we shall attack that problem from a different point of view.)

c) (8 points) Find all eigenvalues $\lambda_1, \lambda_2, \lambda_3$ for M and their corresponding eigenvectors v_1, v_2, v_3. (In our case the characteristic polynomial is of degree three. Even if Cardano already in 1545 presented a general formula for the solution of polynomial equations of degree three, we simplify your problem by telling you that $\lambda = 1$ is an eigenvalue. The corresponding eigenvector is

$$v_1 = \begin{pmatrix} 7 \\ 13 \\ 4 \end{pmatrix}$$

Thus you know one of the solutions to the characteristic equation and by long division of polynomials (or some other method) you should be able to find the two remaining solutions. An example at the end of the sheet illustrates long division of polynomials.

d) (5 points) Compute the determinant $det(M)$ of the matrix M by the general formula. A general result in linear algebra says that the determinant of a matrix equals the product of the eigenvalues. Check this result in our case. (This exercise is a test of your skills in the curriculum, and we will not need this computation for anything else in this assignment.)

e) (4 points) Next we study a more specific example. We consider 240 students and their choices of entering time for the lecture. The distribution between early, during preparation and late for the first lecture is given by $x_1 = 180$ early, $y_1 = 20$ during preparation and $z_1 = 40$ late, i.e.

$$P_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 180 \\ 20 \\ 40 \end{pmatrix}$$

Compute P_2 and P_3. (Notice that we allow non-integer persons, e.g. 109,6 persons)

f) (5 points) To see what happens to the distribution in the long run (many lectures later) we shall use what is called *spectral theory*, i.e. a clever way to exhibit information from the eigenvectors and -values of the matrix M. First we express the initial vector P_1 as a linear combination of eigenvectors, i.e. write

$$P_1 = a_1v_1 + a_2v_2 + a_3v_3$$

as a linear combination of the eigenvectors you found in exercise c), for suitable values of a_1, a_2 and a_3.

g) (8 points) We are going to use the decomposition of P_1 in exercise f) to compute $P_n = M^{n-1}P_1$ for all values of n: Find a formula for P_n using the entries of P_1, the eigenvalues of M and n. When you have established the formula you can use it to describe what happens to the distribution in the long run, i.e. compute the limit of P_n when $n \to \infty$.

h) (5 points) This last exercise is more theoretical than the first seven exercises. The matrix M is special in the sense that the sum of each column equals 1. This implies in fact that 1 is an eigenvalue for M. Can you argue for this statement?

An example to illustrate long division of polynomials.

Let us compute $(x^3-2x^2+4x+7) : (x+1)$. The result should be a polynomial of degree two. We start by the highest power of x, i.e. x^3 and estimates the "quotient" of x^3-2x^2+4x+7 by $x+1$, when restricting ourselves to the highest powers. The answer is x^2. As for ordinary long division we multiply and get

$$\begin{array}{l} x^3 - 2x^2 + 4x + 7 : (x+1) = x^2 \\ \underline{x^3 + \ x^2} \end{array}$$

Subtracting an dpulling down the "next" term gives

$$\begin{array}{l} x^3 - 2x^2 + 4x + 7 : (x+1) = x^2 \\ \underline{x^3 + \ x^2} \\ \quad\ - 3x^2 + 4x \end{array}$$

The "quotient" of $-3x^2+4x$ by $x+1$ is $-3x$. Multiplication of $x+1$ by $-3x$ gives $-3x^2-3x$, this we subtract from $-3x^2+4x$;

$$\begin{array}{l} x^3 - 2x^2 + 4x + 7 : (x+1) = x^2 - 3x \\ \underline{x^3 + \ x^2} \\ \quad\ - 3x^2 + 4x \\ \quad\ \underline{-3x^2 - 3x} \\ \qquad\qquad\quad 7x + 7 \end{array}$$

Finally we see that $x+1$ has to be multiplied by 7 for to get $7x+7$. Thus

$$\begin{array}{l} x^3 - 2x^2 + 4x + 7 : (x+1) = x^2 - 3x + 7 \\ \underline{x^3 + \ x^2} \\ \quad\ - 3x^2 + 4x \\ \quad\ \underline{-3x^2 - 3x} \\ \qquad\qquad\quad 7x + 7 \\ \qquad\qquad\quad \underline{7x + 7} \\ \qquad\qquad\qquad\ \ 0 \end{array}$$