Komplekse tall.

Et komplekst tall er på formen $\boxed{a+bi, \quad a,b\in\mathbb{R},\ i^2=-1.}$

Eks 1

a) $(3+5i)+(2-3i) = (3+2)+(5-3)i = 5+2i$

b) $(3+5i)(2-3i) = 3\cdot 2 + 3\cdot(-3i) + 5i\cdot 2 - \underbrace{3i5i}$ $\left(3\cdot 5\cdot i^2\right)$

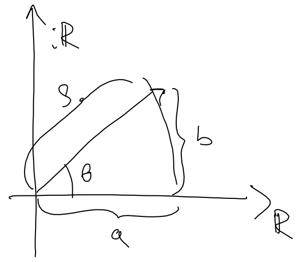
$= 6 - 9i + 10i + 15$

$= 21 + i$

c) $\dfrac{3+5i}{2+3i} = \dfrac{(3+5i)(2-3i)}{(2+3i)(2-3i)} = \dfrac{21+i}{4+9} = \dfrac{21}{13} + \dfrac{1}{13}i$

Geometrisk representasjon

$z = a+bi$



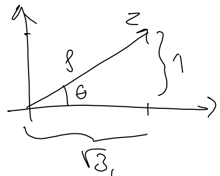
To representasjoner:

1. Kartesisk $z = a+bi$
2. Polarform $z = \rho(\cos\theta + i\sin\theta)$
3. Eksponentialform $z = \rho e^{i\theta}$

(Digresjon:
$z_1, z_2, \quad z_1 = \rho_1 e^{i\theta_1},\ z_2 = \rho_2 e^{i\theta_2} \Rightarrow z_1 z_2 = \rho_1\rho_2 e^{i(\theta_1+\theta_2)}$)

Eks 2: (Kartesisk $\rightarrow$ Polarform)

La $z = \sqrt{3} + i$. Skriv z på polarform. $(z = \rho(\cos\theta + i\sin\theta))$

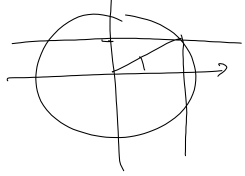


$\rho^2 = (\sqrt{3})^2 + 1^2 = 4 \Rightarrow \rho = 2.$

$\rho\cos\theta = \sqrt{3}, \quad \cos\theta = \frac{\sqrt{3}}{2}$

$\rho\sin\theta = 1, \quad \sin\theta = \frac{1}{2}$ $\Big\}$ $\theta = \frac{\pi}{6}$

$z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ $\left[z = 2e^{i\frac{\pi}{6}}\right]$



De Moivre's formel: $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

$\left[(e^{i\theta})^n = e^{in\theta}\right] \leftarrow$

Eks 4: La $z = \cos\theta + i\sin\theta$. Vis at $z^n + \frac{1}{z^n} = 2\cos(n\theta)$.

$z^n + \frac{1}{z^n} = z^n + \frac{\overline{z^n}}{|z^n|^2} = z^n + \overline{z^n} = 2\mathrm{Re}(z^n) = 2\cos(n\theta)$

$|z| = 1 \qquad z^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$

$\mathrm{Re}(z^n) = \cos(n\theta).$

Differenslikninger

Definisjon: En andre ordens lineær homogen differenslikning er en likning på formen

$$x_{n+2} + b\,x_{n+1} + c\,x_n = 0 \qquad n \geq 0$$

$b, c \in \mathbb{R}$.

Eks 5: (Fibonacci tallene) $x_{n+2} = x_{n+1} + x_n$

$x_0 = x_1 = 1$.

$$x_n = \frac{\sqrt{5}}{5}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right].$$

~~Løsn~~

Hvordan finner man en løsning? Anta $x_n = r^n$

$x_{n+1} = r^{n+1} = r\,r^n$.

$x_{n+2} = r^{n+2} = r^2 r^n$

Setter inn i likningen: $r^2 r^n + b r r^n + c r^n = 0$

$$r^n \underbrace{\left(r^2 + br + c\right)}_{\text{Kar. likn.}} = 0$$

La r_1, r_2 $(r_1 \neq r_2)$ være røtter i den kar. likn.

$x_n^1 = C r_1^n$, $x_n^2 = D r_2^n$, $C, D \in \mathbb{R}$.

$x_n = x_n^1 + x_n^2$. (sjekker at x_n er en løsning.)

$$x_{n+2} + b x_{n+1} + c x_n =$$
$$x_{n+2}^1 + x_{n+2}^2 + b\left(x_{n+1}^1 + x_{n+1}^2\right) + c\left(x_n^1 + x_n^2\right) =$$
$$\underbrace{\left(x_{n+2}^1 + b x_{n+1}^1 + c x_n^1\right)}_{0} + \underbrace{\left(x_{n+2}^2 + b x_{n+1}^2 + c x_n^2\right)}_{0} = 0$$

Dermed er $x_n = x_n^1 + x_n^2$ en løsning.

$$x_n = C r_1^n + D r_2^n.$$

Løsningsmetode:

1. Finn røttene r_1, r_2 til den karakteristiske likningen.
2. Skriv opp løsningen:

<u>Tilfelle 1</u>: $r_1, r_2 \in \mathbb{R}$, $r_1 \neq r_2$.

$$x_n = C r_1^n + D r_2^n \qquad C, D \in \mathbb{R}.$$

<u>Tilfelle 2</u>: $r_1, r_2 \in \mathbb{R}$, $r_1 = r_2$.

$$x_n = C r_1^n + D n r_1^n \qquad C, D \in \mathbb{R}.$$

<u>Tilfelle 3</u>: $r_1, r_2 \in \mathbb{C}$ $(r_2 = \bar{r}_1)$

kompleks form: $x_n = E r_1^n + \bar{E}\bar{r}_1^n$, $E \in \mathbb{C}$.

reell form: $x_n = C\rho^n \cos(n\theta) + D\rho^n \sin(n\theta)$, $C, D \in \mathbb{R}$

$r_1 = \rho e^{i\theta}$.

<u>Eks 6:</u> $x_{n+2} - 2\sqrt{3}\,x_{n+1} + 4x_n = 0$

Finn den generelle løsningen.

1. Kar. likn. $r^2 - 2\sqrt{3}\,r + 4 = 0$

$$r = \frac{2\sqrt{3} \pm \sqrt{4\cdot 3 - 4\cdot 4}}{2} = \sqrt{3} \pm i.$$

2. Tilfelle 3. To komplekse røtter. $r_1 = \sqrt{3} + i$, $r_2 = \sqrt{3} - i$.

Fra eks. 2 $r_1 = 2e^{i\frac{\pi}{6}}$. $\left(\rho = 2,\ \theta = \frac{\pi}{6}\right)$

$$\underline{\underline{x_n = 2^n\left(C\cos\left(n\tfrac{\pi}{6}\right) + D\sin\left(n\tfrac{\pi}{6}\right)\right)}}.$$

Finn en løsning s.a. $x_0 = x_1 = 1$.

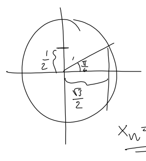
$x_0 = 1 \Rightarrow x_0 = 2^0\left(C\cos(0) + D\sin(0)\right)$

$= C. \Rightarrow \underline{C = 1}$

$x_1 = 1 \Rightarrow x_1 = 2\left(C\cos\left(\tfrac{\pi}{6}\right) + D\sin\left(\tfrac{\pi}{6}\right)\right)$

$= 2\left(C\frac{\sqrt{3}}{2} + D\cdot\frac{1}{2}\right)$

$= C\sqrt{3} + D$



$\Rightarrow 1 = \sqrt{3} + D \quad \underline{D = 1 - \sqrt{3}}$

$$\underline{\underline{x_n = 2^n\left(\cos\left(n\tfrac{\pi}{6}\right) + \left(1-\sqrt{3}\right)\sin\left(n\tfrac{\pi}{6}\right)\right)}}$$

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Inhomogene likninger

$$(*) \quad x_{n+2} + b x_{n+1} + c x_n = f(n), \quad n \geq 0.$$

Løsningsmetode: (Generell løsning)

1. Finn den generelle løsningen av homogen likn. (x_n^h)
2. Finn én løsning av inhomogen likn. (x_n^p)

Den generelle løsningen av (*)

$$x_n = x_n^h + x_n^p.$$

Problemet er punkt 2.

Anta f(n) er et polynom in. $\left(f(n) = 2n^2 + 1\right)$

$$f(n) = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_0$$

Vi gjetter på

$$x_n^p = A_k n^k + \cdots + A_0$$

NB: For hver rot i den kar. likn. som er 1 går vi opp en grad.

Eks. 7 $\quad x_{n+2} - 6x_{n+1} + 8x_n = 9n + 3$

1. Kar. likn. $r^2 - 6r + 8 = 0$

$$r = \frac{6 \pm \sqrt{36 - 4 \cdot 8}}{2} = \frac{6 \pm 2}{2} = 3 \pm 1.$$

$r_1 = 2, \; r_2 = 4.$

$$x_n^h = C 2^n + D 4^n$$

2. Anta $x_n^p = An + B$.

$x_{n+2}^p = An + 2A + B$

$x_{n+1}^p = An + A + B$

Setter inn i likn.

$$x_{n+2}^p - 6x_{n+1}^p + 8x_n^p = 9n + 3$$

$$3An + (-4) \cdot A + 3B = 9n + 3 \qquad n \geq 0$$

$$\begin{cases} 3A = 9 & A = 3, \\ 3B - 4A = 3 & 3B = 3 + 4 \cdot 3, \; B = 5. \end{cases}$$

$$x_n^p = 3n + 5.$$

$$x_n = x_n^h + x_n^p = C 2^n + D 4^n + 3n + 5.$$