Plenum uke 39.

Oppgaver: A7: 3, 4, 6, 11, 12, 14.
A8: 1.
B2: 6, 10, 11.

7.3 Regn ut:

a) $i(i-1) = i^2 - i = \underline{\underline{-1-i}}$

b) $(2+i)(2+i) = 2^2 + 2\cdot 2i + i^2$
$= 4 + 4i - 1$
$\underline{\underline{= 3 + 4i}}$

c) $(4-3i)(4+5i) = 4^2 - 4\cdot 3i + 4\cdot 5i + (-3i)(5i)$
$= 16 - 12i + 20i - 15i^2$
$= 16 + 15 + 8i$
$\underline{\underline{= 31 + 8i}}$

d) $i^3 = i^2 i = \underline{\underline{-i}}$

e) $i^4 = i^2 i^2 = -1 \cdot -1 = \underline{1}$

f) $(3-2i)(3+2i) = 3^2 - (2i)^2 = 9 + 4 = \underline{\underline{13}}$

7.4 Regn ut:

a) $$\frac{4-8i}{1-i} = \frac{(4-8i)(1+i)}{(1-i)(1+i)} = \frac{4-8i+4i+8}{1-i^2}$$

$$= \frac{12-4i}{1+1} = \underline{\underline{6-2i}}$$

b) $$\frac{1}{2+4i} = \frac{1\cdot(2-4i)}{2^2+4^2} = \frac{2}{20} - \frac{4}{20}i$$

$$= \underline{\underline{\frac{1}{10} - \frac{1}{5}i}}$$

c) $$\frac{3+16i}{2-6i} = \frac{(3+16i)(2+6i)}{(2-6i)(2+6i)}$$

$$= \frac{6 + 32i + 18i - \overbrace{16\cdot 6}^{60+36=96}}{2^2+6^2}$$

$$= \frac{-90}{40} + \frac{50}{40}i = \underline{\underline{-\frac{9}{4} + \frac{5}{4}i}}$$

d) $\frac{1+5i}{5-i} = \frac{(1+5i)(5+i)}{5^2+1}$

$= \frac{\cancel{5} + i + 25i \cancel{- 5}}{26} = \frac{26}{26}i = \underline{\underline{i}}$

e) $\frac{i}{6+7i} = \frac{i(6-7i)}{6^2+7^2} = \underline{\underline{\frac{7}{85} + \frac{6}{85}i}}$

$36+49 = 85$

f) $\frac{i^2}{i+1} = \frac{-1(1-i)}{2} = \frac{-1+i}{2} = \underline{\underline{-\frac{1}{2} + \frac{1}{2}i}}$

7.6 Løs andregradslikningene:

a) $z^2+1=0$ $\quad (z+i)(z-i) = z^2 - i^2 = z^2+1$

Dermed har vi løsningene $\{-i, i\}$

b) $z^2-2z+4=0$ Bruker a-b-c formel:

$z = \frac{2 \pm \sqrt{2^2-4 \cdot 1 \cdot 4}}{2} = \frac{2 \pm \sqrt{-12}}{2}$

$= 1 \pm \sqrt{-3} = 1 \pm \sqrt{3}\sqrt{-1} = \underline{\underline{1 \pm \sqrt{3}i}}$

Vi har løsningene $\{1+\sqrt{3}i, 1-\sqrt{3}i\}$

Dersom vi vil sjekke kan vi
sette inn:

$$\underbrace{(1+\sqrt{3}i)^2 - 2(1+\sqrt{3}i) + 4}\overset{?}{=} 0$$

$$1 + \cancel{2\sqrt{3}i} - 3 - 2 - \cancel{2\sqrt{3}i} + 4$$

$$= 1 - 3 - 2 + 4 = 0 \text{ ! Riktig.}$$

Hvordan vet vi at $(1-\sqrt{3}i)$ er en
løsning?

Alternativ sjekk:

$$(z-(1+\sqrt{3}i))(z-(1-\sqrt{3}i)) \overset{?}{=} z^2-2z+4.$$

c) $z^2+2z+5=0.$

$$z = \frac{-2 \pm \sqrt{2^2-4\cdot 1\cdot 5}}{2} = -1 \pm \frac{1}{2}\sqrt{-16}$$

$$= -1 \pm \sqrt{-4}$$

$$\underline{\underline{= -1 \pm 2i}}$$

$$z \in \{-1+2i, -1-2i\}$$

d) $3z^2 - 4z + 2 = 0$

$4 \cdot 4 - 4 \cdot 6 = 4 \cdot (4 - 6) = -8$

$$z = \frac{4 \pm \sqrt{4^2 - 4 \cdot 3 \cdot 2}}{2 \cdot 3} = \frac{4 \pm \sqrt{-8}}{2 \cdot 3}$$

$$= \frac{2}{3} \pm \frac{1}{3}\sqrt{\frac{8}{4}}\, i = \underline{\underline{\frac{2}{3} \pm \frac{\sqrt{2}}{3} i}}$$

$$z \in \left\{ \frac{2}{3} + \frac{\sqrt{2}}{3} i,\ \frac{2}{3} - \frac{\sqrt{2}}{3} i \right\}$$

7.11 Skriv tallene på polarform.

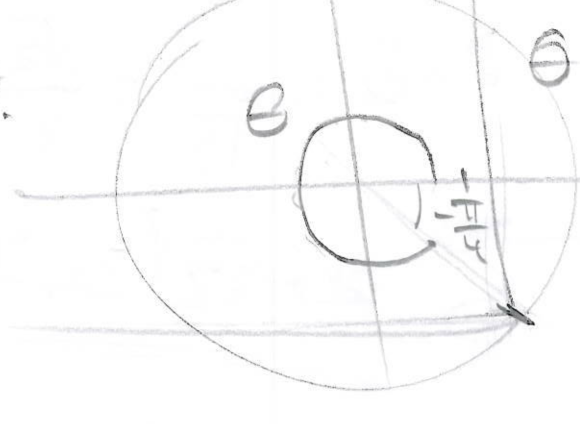
a) $z = 2 - 2i$. $\quad \boxed{z = \rho \cos\Theta + i \rho \sin\Theta}$

$$\rho = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}.$$

$$\rho \cos\Theta = 2 \iff \cos\Theta = \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2}$$

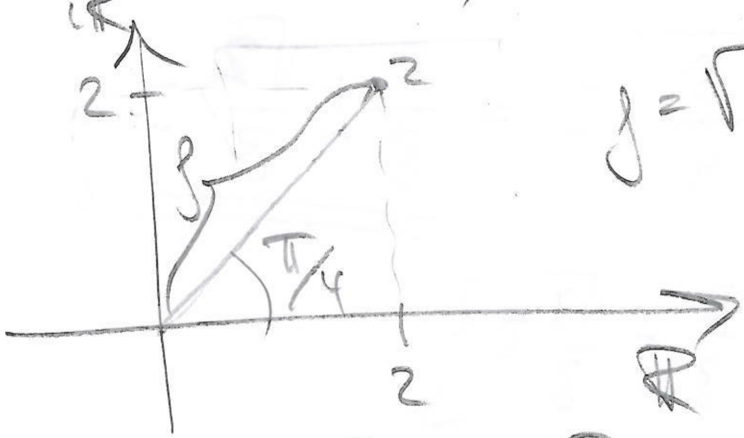
$$\rho \sin\Theta = -2 \iff \sin\Theta = -\frac{2}{2\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

Må finne Θ. $\quad \Theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$



$$z = 2\sqrt{2}\left(\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right)$$

b) $z = 2 + 2i$, $\quad z = \rho\cos\theta + \rho i \sin\theta$



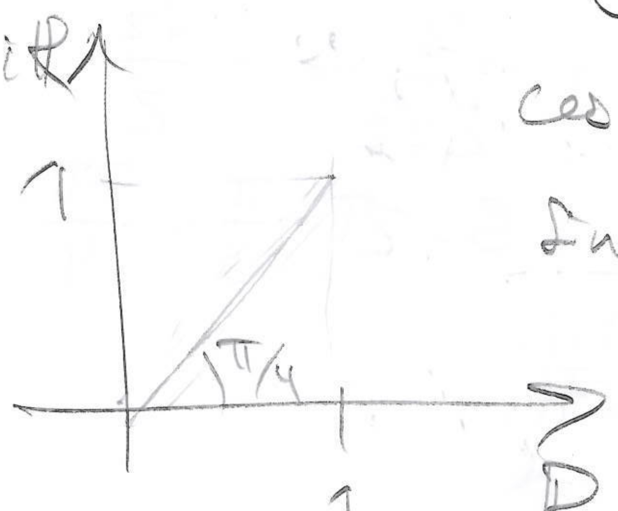
$$\rho = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$$

$$\cos\theta = \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2}, \quad \sin\theta = \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\Longrightarrow \theta = \frac{\pi}{4}$$

$$z = 2\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$$

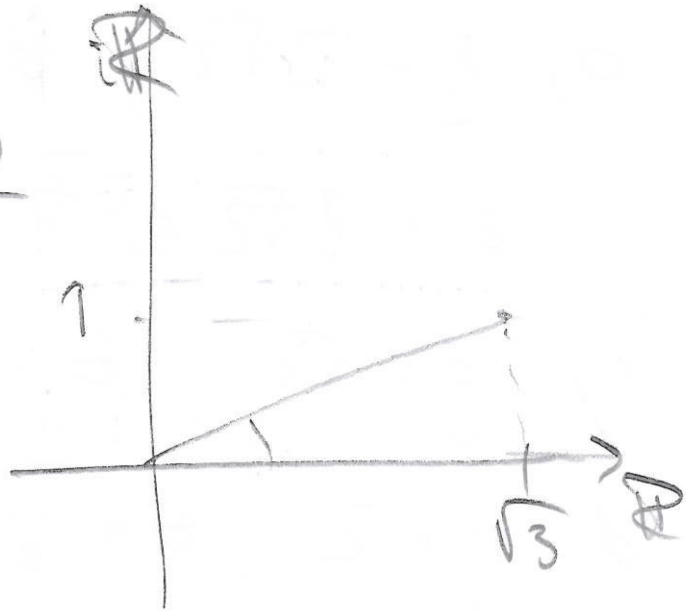
c) $z = 1 + i$, $\quad \rho = \sqrt{1^2 + 1^2} = \sqrt{2}$



$$\left.\begin{aligned} \cos\theta &= \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\ \sin\theta &= \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \end{aligned}\right\} \Rightarrow \theta = \frac{\pi}{4}$$

$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

d) $z = \sqrt{3} + i$



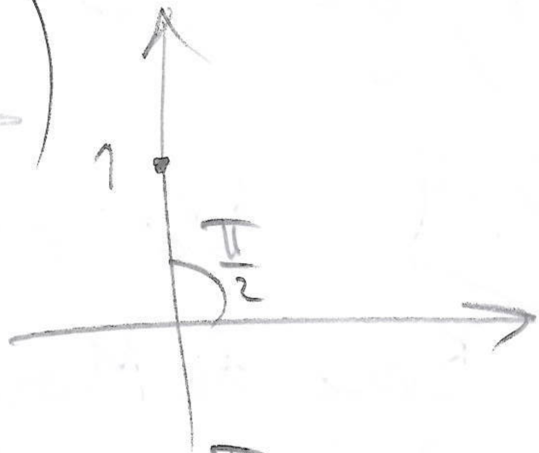
$$\rho = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$$

$$\cos\theta = \frac{\sqrt{3}}{2}, \quad \sin\theta = \frac{1}{2}$$

$$\Rightarrow \theta = \frac{\pi}{6}$$

$$\underline{z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)}$$

e) $z = i$



$$\rho = \sqrt{1^2} = 1.$$

$$\cos\theta = 0, \quad \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$$

$$\underline{z = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)}$$

<u>7.12</u> Skriv tallene på oppgave 7.4 på eksponentialform.

$$\boxed{z = \rho e^{i\theta}}$$

a) $\rho = 2\sqrt{2}$, $\theta = \frac{2\pi}{4}$

$$\Rightarrow z = 2\sqrt{2}\, e^{i\frac{2\pi}{4}}$$

b) $\rho = 2\sqrt{2}$, $\theta = \frac{\pi}{4}$

$z = 2\sqrt{2}\, e^{i\frac{\pi}{4}}$

c) $\rho = \sqrt{2}$, $\theta = \frac{\pi}{4} \Rightarrow z = \sqrt{2}\, e^{i\frac{\pi}{4}}$

d) $\rho = 2$, $\theta = \frac{\pi}{6} \Rightarrow z = 2e^{i\frac{\pi}{6}}$

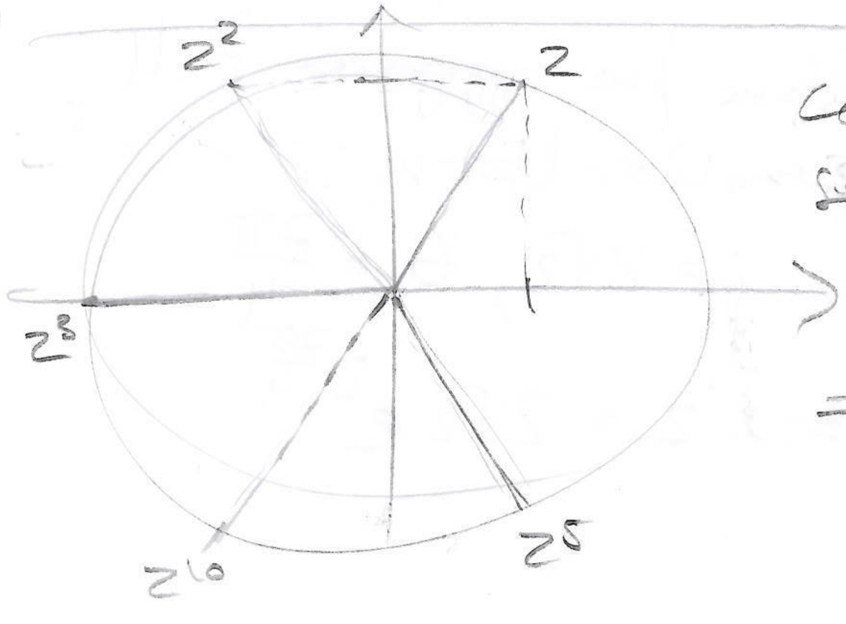
e) $\rho = 1$, $\theta = \frac{\pi}{2} \Rightarrow z = e^{i\frac{\pi}{2}}$.

7.14 La $z = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$.

Bruk de Moivres formel og finn z^2, z^3, z^5 og z^{10}.

de Moivres formel:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$



$$\left.\begin{aligned}\cos\frac{\pi}{3} &= \frac{1}{2}\\ \sin\frac{\pi}{3} &= \frac{\sqrt{3}}{2}\end{aligned}\right\}$$

$$\Rightarrow z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

de Moivre $\Rightarrow z^2 = \cos\frac{2\pi}{3} + i\sin\left(\frac{2\pi}{3}\right)$

$$\left.\begin{array}{l} \cos\frac{2\pi}{3} = -\frac{1}{2} \\ \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} \end{array}\right\} z^2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

$$z^3 = \cos\pi + i\sin\pi.$$

$$\left.\begin{array}{l} \cos\pi = -1 \\ \sin\pi = 0 \end{array}\right\} \Rightarrow z^3 = -1$$

$$z^5 = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$$

$$\left.\begin{array}{l} \cos\frac{5\pi}{3} = \frac{1}{2} \\ \sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2} \end{array}\right\} \Rightarrow z^5 = \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

$$z^{10} = \cos\frac{10\pi}{3} + i\sin\frac{10\pi}{3}$$

$$\cos\frac{10\pi}{3} = \cos\left(\frac{10\pi}{3} - 2\pi\right) = \cos\frac{4\pi}{3} = -\frac{1}{2}$$

$$\sin\frac{10\pi}{3} = \sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$$

$$z^{10} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Multiplikasjon med z gir rotasjon om origo med vinkel $\frac{\pi}{3}$.

$z^{10} = z^4$, $z^5 = \bar{z}$, $z^2 = -z^5$

$z^{10} = -z$. (Mange mulige obs.)

8.1 Finn løsningen til differenslikningen

$$x_{n+2} - 5x_{n+1} + 4x_n = 0, \quad n \geq 0$$

$x_0 = 1$, $x_1 = -2$.

Karakteristisk likning $r^2 - 5r + 4 = 0$

$$r = \frac{5 \pm \sqrt{5^2 - 4 \cdot 1 \cdot 4}}{2} = \frac{5 \pm 3}{2}$$

$r_1 = 4$, $r_2 = 1$. (To ulike reelle røtter.)

Den generelle løsningen er på formen

$$x_n = C4^n + D1^n, \quad n \geq 0$$

$C, D \in \mathbb{R}$.

$$\left.\begin{aligned} x_0 = 1 &\Longrightarrow 1 = C + D \\ x_1 = -2 &\Longrightarrow -2 = 4C + D \end{aligned}\right\} \begin{aligned} &: \mathrm{I} \\ &: \mathrm{II} \end{aligned}$$

(Innsettingsmetoden)

I $D = 1 - C \quad \mathrm{II} \Longrightarrow -2 = 4C + (1 - C)$

$$-2 = 1 + 3C \Longrightarrow \underline{C = -1}$$

Dermed får vi løsningen: $D = 1-(-1) = 2.$

$$\underline{\underline{x_n = -4^n + 2}}$$

B2.6 La $z = 1 + i$ og $w = 2 - i$.

Skriv $\frac{z}{w}$ og $\bar{z} - \frac{w}{z}$ på kartesisk form.

$$\frac{z}{w} = \frac{1+i}{2-i} = \frac{(1+i)(2+i)}{(2-i)(2+i)} = \frac{2+3i-1}{4+1}$$

$$= \underline{\underline{\frac{1}{5} + \frac{3}{5}i}}$$

$\bar{z} - \frac{w}{2} = \ldots$

$$\frac{w}{2} = \frac{2-i}{1+i} = \frac{(2-i)(1-i)}{1+1}$$

$$= \frac{2-3i-1}{2} = \frac{1}{2} - \frac{3}{2}i$$

$$\bar{z} - \frac{w}{2} = 1 - i - \left(\frac{1}{2} - \frac{3}{2}i\right)$$

$$= \underline{\underline{\frac{1}{2} + \frac{1}{2}i}}$$

B 2.10 Hvilket av følgende uttrykk er en forenkling av uttrykket

$$z = \frac{1}{\sqrt{2}}(1+i)\cdot e^{-i\frac{\pi}{4}}$$

a) 1

b) $\frac{1}{\sqrt{2}}$

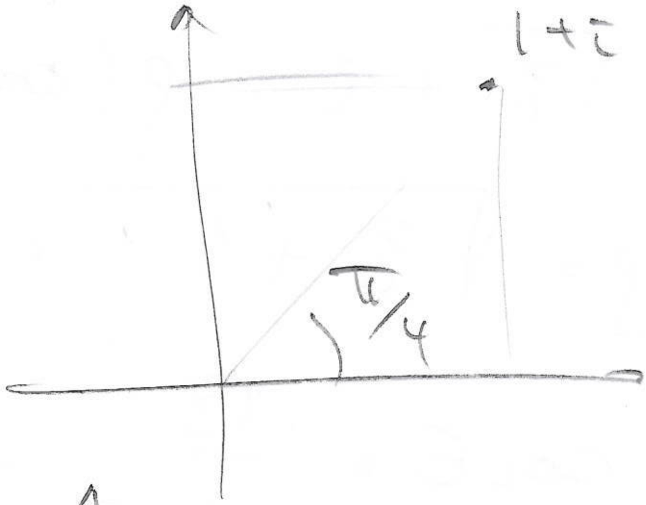
c) $1-i$

$$1+i = \rho(\cos\theta + i\sin\theta)$$

$$\rho = \sqrt{1^2+1^2} = \sqrt{2}.$$

$$\cos\theta = \frac{1}{\sqrt{2}}, \quad \sin\theta = \frac{1}{\sqrt{2}}.$$

$1+i = \sqrt{2}\, e^{i\frac{\pi}{4}}$



$$z = \frac{1}{\cancel{\sqrt{2}}} \cancel{\sqrt{2}}\, e^{i\frac{\pi}{4}} e^{-i\frac{\pi}{4}}$$

$$= e^{i\left(\frac{\pi}{4}-\frac{\pi}{4}\right)} = e^{i0} = \underline{\underline{1}}$$

Andre måter å finne ut av dette på?

$$|z| = \frac{1}{\sqrt{2}}\,|1+i|\,\left|e^{-i\frac{\pi}{4}}\right| = \frac{1}{\sqrt{2}}\,|1+i|$$

$= 1$ (Hva er modulus til b og c?)

<u>B 2.11</u> Hvilket av følgende uttrykk er en forenkling av

$$z = \frac{1}{2}\left(\sqrt{3}+i\right)\cdot e^{i\frac{\pi}{3}} \quad ?$$

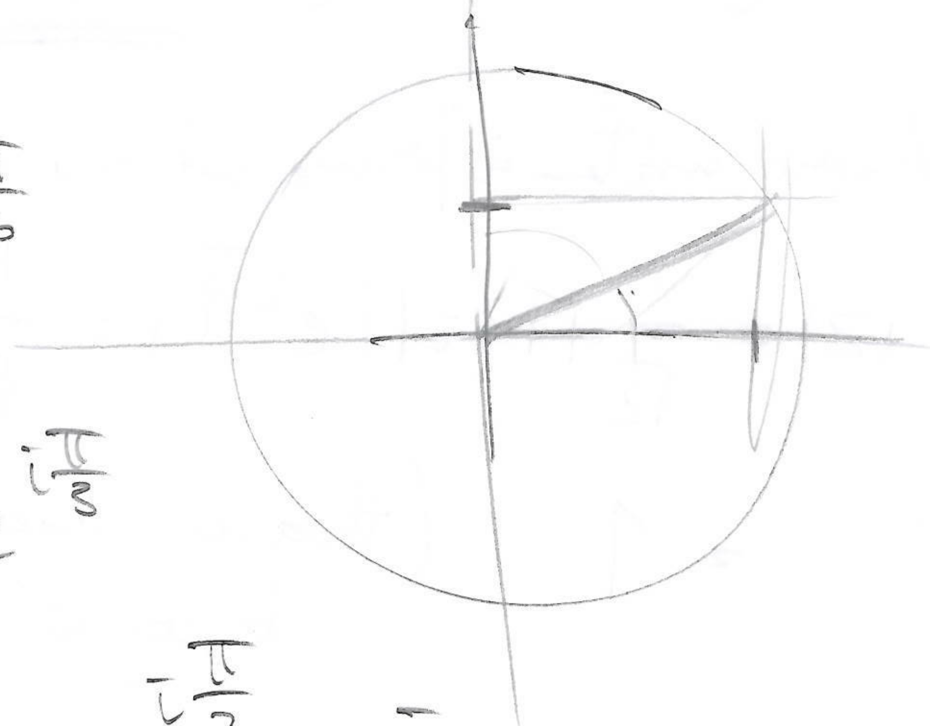
a) i b) 1 c) $\frac{1}{2}\sqrt{3}+2i$

$$\sqrt{3} + i = \rho(\cos\theta + i\sin\theta)$$

$$\rho = \sqrt{3+1} = 2.$$

$$\cos\theta = \frac{\sqrt{3}}{2}, \quad \sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}$$



$$\sqrt{3} + i = 2e^{i\frac{\pi}{6}}$$

$$z = \frac{1}{2} 2e^{i\frac{\pi}{6}} e^{i\frac{\pi}{3}}$$

$$= e^{i\left(\frac{\pi}{6}+\frac{\pi}{3}\right)} = e^{i\frac{\pi}{2}} = \underline{\underline{i}}$$

<u>Svar: a</u>