Plenum uke 45

Oppgaver: A11: 8, 9, 10

A12: 2, 3, 5

B3: 11, 12.

11.8 Finn de ubestemte integralene

a) $\int (x^5 - 3x)\,dx = \frac{1}{6}x^6 - \frac{3}{2}x^2 + C.$

b) $\int x^{5/6} - 1\,dx = \frac{1}{\frac{5}{6}+1} x^{\frac{5}{6}+1} - x + C$

$\left(\frac{5}{6}+1 = \frac{11}{6}\right)$ $= \frac{6}{11}x^{\frac{11}{6}} - x + C$

c) $\int 3xe^{x^2}\,dx = \frac{3}{2}\int 2xe^{x^2}\,dx$

$u(x) = x^2$

$u'(x) = 2x$

$= \frac{3}{2}\int e^u\,du$

$= \frac{3}{2}e^{x^2} + C.$

d) $\int x^3 \sin(x^4)\,dx = I$

$u(x) = x^4, \quad u'(x) = 4x^3 \qquad du = u'(x)dx$

$$I = \frac{1}{4}\int \sin(\underbrace{x^4}_{u})\underbrace{4x^3dx}_{du}$$

$$= \frac{1}{4}\int \sin(u)\,du = \frac{-1}{4}\cos(x^4) + C$$

e) $\int \frac{3}{x(x-2)}\,dx = I$

Bruker delbrøksoppspalting:

$$\frac{3}{x(x-2)} = \frac{A}{x} + \frac{B}{(x-2)}$$ (Er dette alltid mulig?)

$$3 = A(x-2) + Bx \quad (\forall x \in \mathbb{R})$$

(Observer at vi har en likning for hver x! Uendelig mange.)

$x = 2 \Rightarrow 3 = 2B \Rightarrow B = \frac{3}{2}$

$x = 0 \Rightarrow 3 = -2\cancel{B}A \Rightarrow A = -\frac{3}{2}$

$$x = 1 \Rightarrow 1 = 3A \Rightarrow A = \frac{1}{3}$$

$$I = \frac{1}{3}\int \frac{1}{x-1}\,dx - \frac{1}{3}\int \frac{1}{x+2}\,dx$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + C$$

$$= \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C$$

11.9 Finn de ubestemte integralene.

a) $$\int \underset{u'}{e^x} \underset{v}{\sin x}\,dx = I_1 \qquad \begin{array}{l} u = e^x \\ v = \sin x \end{array}$$

$$\int u'v = uv - \int uv'$$

$$I_1 = e^x \sin x - \underbrace{\int \overset{u'}{e^x} \overset{v}{\cos x}\,dx}_{I_2}$$

$$I_2 = e^x \cos x + \underbrace{\int e^x \sin x\,dx}_{I_1}$$

$$\frac{3}{x(x-2)} = -\frac{3}{2x} + \frac{3}{2(x-2)}$$

$$I = -\int \frac{3}{2x}\,dx + \int \frac{3}{2(x-2)}\,dx$$

$$= -\frac{3}{2}\ln|x| + \frac{3}{2}\ln|x-2| + C$$

$$= \frac{3}{2}\left(\ln|x-2| - \ln|x|\right) + C$$

$$= \frac{3}{2}\ln\left|\frac{x-2}{x}\right| + C$$

[Hva kaller vi her?]

f) $$\int \frac{1}{(x-1)(x+2)}\,dx = I$$

Implikasjonen holder kun for

$$\frac{1}{(x-1)(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+2)}$$

$$\Rightarrow 1 = A(x+2) + B(x-1) \quad \begin{pmatrix} x \neq 1 \\ x \neq -2 \end{pmatrix}$$

$$x = -2 \Rightarrow 1 = B(-2-1) = -3B$$

$$B = -\frac{1}{3}$$

$$I_1 = e^x \sin x - e^x \cos x - I_1$$

$$\Rightarrow 2I_1 = e^x(\sin x - \cos x)$$

$$I_1 = \frac{1}{2}e^x(\sin x - \cos x) + C$$

Alternativ måde: Husk

$$e^{ix} = \cos x + i \sin x \Rightarrow$$

$$\underbrace{\int e^x \cdot e^{ix}\,dx}_{I_3} = \int e^x \cos x\,dx + \underbrace{i\int e^x \sin x\,dx}_{iI_1}$$

$$\Rightarrow \boxed{I_1 = \mathrm{Im}(I_3)}$$

$$I_3 = \int e^{(1+i)x}\,dx = \frac{1}{1+i}e^{(1+i)x} + C$$

$$\frac{1}{1+i} = \frac{1-i}{(1+i)(1-i)} = \frac{1-i}{2} = \frac{1}{2} - \frac{1}{2}i$$

$$e^{(1+i)x} = e^x(\cos x + i \sin x)$$

$$\frac{1}{1+i}e^{(1+i)x} = \frac{1}{2}(1-i)e^{x}(\cos x + i\sin x)$$

$$= \frac{1}{2}e^{x}\Big[(\cos x + \sin x)$$

$$+ i(-\cos x + \sin x)\Big]$$

Altså er

$$I_3 = \frac{1}{2}e^{x}\Big[(\cos x + \sin x) + i(\sin x - \cos x)\Big]$$

$$+ C \longleftarrow \text{kompleks}$$

$\Longrightarrow$

$$\mathrm{Im}(I_3) = \frac{1}{2}e^{x}(\sin x - \cos x) + C$$

"ny konstant"

b) $\displaystyle\int \frac{\ln x}{x}\,dx$ $\qquad u(x) = \ln x$

$\qquad\qquad\qquad\qquad u'(x) = \frac{1}{x}$

$$\int \frac{\ln x}{x}\,dx = \int u(x)\underbrace{u'(x)\,dx}_{du}$$

$$= \int u\,du = \frac{1}{2}u^2 + C$$

$$\int \frac{\ln x}{x}\,dx = \frac{1}{2}(\ln x)^2 + C$$

c/ $\int \frac{1}{x^2-4}\,dx$

Husk: $x^2-4 = (x-2)(x+2)$

$$\frac{1}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}$$

$$1 = A(x+2) + B(x-2)$$

$x = 2 \Rightarrow 1 = 4A \Rightarrow A = \frac{1}{4}$

$x = -2 \Rightarrow 1 = -4B = B = -\frac{1}{4}$.

$$\int \frac{1}{x^2-4}\,dx = \frac{1}{4}\int \frac{1}{x+2}\,dx - \frac{1}{4}\int \frac{1}{x-2}\,dx$$

$$= \frac{1}{4}\left(\ln|x+2| - \ln|x-2|\right) + C$$

$$= \frac{1}{4}\ln\left|\frac{x+2}{x-2}\right| + C$$

11.10 Finn de ubestemte integralene.

a) $\int 3 \ln x \, dx$

Bruker delvis integrasjon.

$u'(x) = 3 \quad v(x) = \ln x$

$u(x) = 3x \quad v'(x) = \frac{1}{x}$

$\int u'v = uv - \int uv'$

$\int 3 \ln x \, dx = 3x \ln x - \int 3x \cdot \frac{1}{x} dx$

$= 3x \ln x - 3x + C$

$= 3x(\ln x - 1) + C$

Sjekk:

$[3x(\ln x - 1)]' = 3(\ln x - 1) + 3x \cdot \frac{1}{x}$

$= 3 \ln x \cancel{-3 + 3}$ ✓

b) $\int \cos^2 x \, dx$

$$\left.\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ 1 &= \cos^2 x + \sin^2 x \end{aligned}\right\}$$

$$\Rightarrow (1 + \cos 2x) = 2\cos^2 x$$

$$\int \cos^2 x \, dx = \frac{1}{2} \int 1 + \cos 2x \, dx$$

$$= \underline{\frac{1}{2} x + \frac{1}{4} \sin 2x + C}$$

$(\sin 2x = 2 \sin x \cos x)$

Alternativt kan man bruke delvis int.
(2 ganger som i 11.9 a)

c) $\int x \ln(x^2) \, dx$ $\left(\begin{array}{l} \text{Fra a har vi} \\ \int \ln x \, dx = x \ln x - x + C \end{array} \right.$

$$= \frac{1}{2} \int \ln(x^2) \, 2x \, dx$$

$$= \frac{1}{2} \int \ln(u) \, du = \frac{1}{2}(u \ln u - u) + C$$

$$= \frac{1}{2}\left(x^2 \ln(x^2) - x^2\right) + C$$

Sjekk:

$$\left(\frac{1}{2}\left(x^2 \ln x^2 - x^2\right)\right)' = \frac{1}{2} 2x \ln x^2 + \frac{1}{\cancel{2}} \cancel{x^2} \cdot \frac{1}{\cancel{x^2}} 2x - x$$

(Feil: poeng)

$$= x \ln x^2 + x - x$$

d) $\int x \sin(\sqrt{x})\,dx$

$u(x) = \sqrt{x}$

$u'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2u}$

$= \int u^2 \sin(u)\,dx$

$u'(x)\,dx = du$

$= \int u^2 \sin(u) 2u\,du$

$\frac{1}{2u}\,dx = du$

$\boxed{dx = 2u\,du}$

$= 2 \underbrace{\int u^3 \sin(u)\,du}_{I}$

Bruker delvis integrasjon.

$$f(u) = u^3 \quad g'(u) = \sin u$$
$$f'(u) = 3u^2 \quad g(u) = -\cos(u)$$
$$\int f g' = f g - \int f' g$$
$$\int u^3 \sin u \, du = -u^3 \cos u + 3 \underbrace{\int u^2 \cos u \, du}_{I_2}$$
$$f(u) = u^2 \quad g'(u) = \cos u$$
$$f'(u) = 2u \quad g(u) = \sin u$$
$$\int u^2 \cos u \, du = u^2 \sin u - 2 \underbrace{\int u \sin u \, du}_{I_3}$$
$$f(u) = u \quad g'(u) = \sin u$$
$$f'(u) = 1 \quad g(u) = -\cos u$$
$$\int u \sin u \, du = -u \cos u + \int \cos u \, du$$
$$= -u \cos u + \sin u$$

Så må vi nøste oss tilbake igjen...

$$\int x \sin\sqrt{x}\, dx = 2I_1$$

$$= -2u^3 \cos u + 6I_2$$

$$= -2u^3 \cos u + 6u^2 \sin u - 12I_3$$

$$= -2u^3 \cos u + 6u^2 \sin u$$

$$+ 12u \cos u + 12 \sin u$$

$$= \underline{\underline{-2x^{3/2} \cos\sqrt{x} + 6x \sin\sqrt{x}}}$$

$$\underline{\underline{+ 12\sqrt{x} \cos\sqrt{x} + 12 \sin\sqrt{x} + C}}.$$

c/ $\int \underbrace{(x^2+x)}_{u(x)} \underbrace{\sin x}_{v'(x)}\, dx$

$u(x) = x^2 + x \qquad v'(x) = \sin x$

$u'(x) = 2x + 1 \qquad v(x) = -\cos x$

$$\int (x^2+x) \sin x\, dx = -(x^2+x)\cos x$$

$$+ \int (2x+1) \cos x\, dx$$

$$\int (2x+1)\cos x \, dx$$

$$u(x) = 2x+1 \qquad v'(x) = \cos x$$

$$u'(x) = 2 \qquad v(x) = \sin x$$

$$\int (2x+1)\cos x \, dx = (2x+1)\sin x \underbrace{- 2\int \sin x \, dx}_{+2\cos x}$$

$$\int (x^2+x)\sin x \, dx = -(x^2+x)\cos x + (2x+1)\sin x + 2\cos x + C$$

$$= (-x^2 - x + 2)\cos x + (2x+1)\sin x + C.$$

12.2 Finn alle løsningene til diff.likningen:

a) $y' + 2xy = x$

(Denne er på formen: $y' + f(x)y = g(x)$
$\Rightarrow F(x) = x^2 \quad F'(x) = f(x) = 2x$)

$$e^{x^2}y' + 2xe^{x^2}y = xe^{x^2}$$

$$\left(e^{x^2}y\right)' = xe^{x^2}$$

$$e^{x^2}y = \int xe^{x^2}\,dx + C$$

$$\left(\int xe^{x^2}\,dx = \frac{1}{2}\int e^{x^2}\underbrace{2x\,dx}_{du} = \frac{1}{2}e^{x^2} + C\right)$$

$$e^{x^2}y = \frac{1}{2}e^{x^2} + C \Rightarrow \underline{\underline{y = \frac{1}{2} + Ce^{-x^2}}}$$

b) $y' + y = e^x$ $\quad \left(f(x) = 1 \right)$

Integrerende faktor er gitt ved

$$e^{F(x)}$$

hvor $F'(x) = f(x)$

$$e^x y' + e^x y = (e^x)^2$$

$$(e^x y)' = e^{2x}$$

$$e^x y = \int e^{2x}\,dx + C$$

$$e^x y = \frac{1}{2} e^{2x} + C$$

$$\underline{\underline{y = \frac{1}{2} e^x + C e^{-x}}}$$

c) $y' + 2y = x e^x$ $\quad \left(\begin{array}{l} f(x) = 2 \\ F(x) = 2x \end{array} \right)$

$$e^{2x} y' + 2 e^{2x} y = e^{2x} x e^x$$

$$(e^{2x} y)' = x e^{3x}$$

$$e^{2x} y = \int x e^{3x}\,dx + C$$

Vi må finne antiderivert til xe^{3x}

Bruker delvis integrasjon.

$$\boxed{\begin{array}{ll} u(x) = x & v'(x) = e^{3x} \\ u'(x) = 1 & v(x) = \frac{1}{3}e^{3x} \\ \int uv' = uv - \int u'v & \end{array}}$$

$$\int x e^{3x} dx = x\frac{1}{3}e^{3x} - \frac{1}{3}\int e^{3x} dx$$

$$= \underbrace{x\frac{1}{3}e^{3x} - \frac{1}{9}e^{3x}}_{\frac{1}{3}\left(x-\frac{1}{3}\right)e^{3x}} + C$$

$$e^{2x}y = \frac{1}{3}\left(x - \frac{1}{3}\right)e^{3x} + C$$

$$\underline{\underline{y(x) = \frac{1}{3}\left(x - \frac{1}{3}\right)e^{x} + Ce^{-2x}}}$$

d) $(x^2+1)y' + 3xy = 6x$

Deler på x^2+1 (Husk $x^2+1 \neq 0$)

$$y' + \underbrace{\frac{3x}{x^2+1}}_{f(x)} y = \underbrace{\frac{6x}{x^2+1}}_{g(x)}$$

$$F(x) = \int f(x)\,dx = \int \frac{3x}{x^2+1}\,dx$$

$$= \frac{3}{2}\int \frac{\overset{=u'}{2x}}{\underset{u=}{x^2}+1}\,dx$$

$$= \frac{3}{2}\int \frac{1}{u+1}\,du$$

$$= \frac{3}{2}\ln|x^2+1| + C$$

Kan velge $C=0$ siden vi kun har behov for $F'(x)=f(x)$ (som holder for alle C)

Dermed har vi integrerende faktor

$$e^{F(x)} = \exp\left(\frac{3}{2}\ln|x^2+1|\right)$$

$$= \left(e^{\ln(x^2+1)}\right)^{\frac{3}{2}} = (x^2+1)^{3/2}$$

$$(x^2+1)^{3/2}y' + 3x\left(\sqrt{x^2+1}\right)y = 6x\sqrt{x^2+1}$$

$$\left((x^2+1)^{3/2}y\right)' = 6x\sqrt{x^2+1}$$

$$(x^2+1)^{3/2}y = 3\int \underbrace{2x}_{u'(x)}\underbrace{\sqrt{x^2+1}}_{u(x)}\,dx$$

$$= 3\int \sqrt{u}\,du = 3\cdot\frac{2}{3}u^{3/2} + C$$

$$= 2(x^2+1)^{3/2} + C$$

$$\underline{\underline{y(x) = 2 + \frac{C}{(x^2+1)^{3/2}}}}$$

D. defineret for alle $x \in \mathbb{R}$.

12.3 Finn alle løsningene til diff likningen

$$xy' + (2x-3)y = 4x^4$$

Anta $x \neq 0$. Deler på x.

$$y' + \left(2 - \frac{3}{x}\right)y = 4x^3 \qquad (*)$$

Finner integrerende faktor:

$$\int 2 - \frac{3}{x}\,dx = \underbrace{2x - 3\ln|x|} + C$$

Velger $C = 0$.

$$e^{2x - 3\ln|x|} = e^{2x}\left(e^{\ln|x|}\right)^{-3}$$

$$= e^{2x} \cdot \frac{1}{|x|^3}$$

Siden

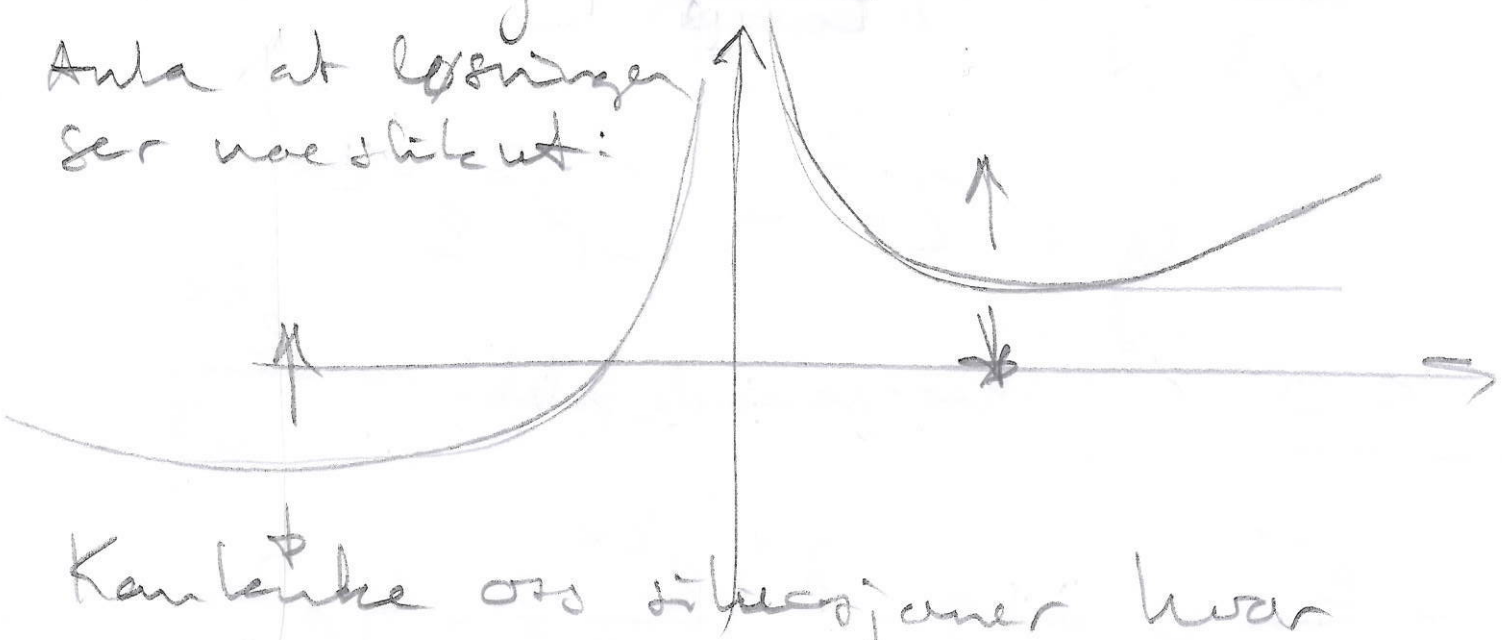
$$\frac{1}{|x|^3} = \begin{cases} \frac{1}{x^3} & x > 0 \\ -\frac{1}{x^3} & x < 0 \end{cases}$$

og $(*)$ ikke er definert i 0

Løser vi likningen separat for

$x < 0$ og $x > 0$

Anta at løsningen ser noe slikt ut:



Kan tenke oss situasjoner hvor en løsning av problemet har forskjellige konstanter for $x > 0$ og $x < 0$.

<u>$x > 0$</u> : Multiplikasjon med int. faktor gir:

$$\frac{e^{2x}}{x^3} y' + \left(2 - \frac{3}{x}\right) \frac{e^{2x}}{x^3} y = 4e^{2x}$$

$$\left(\frac{e^{2x}}{x^3} y\right)' = 4e^{2x}$$

Sjekk:

$$\left(\frac{e^{2x}}{x^3} y\right)' = \frac{e^{2x}}{x^3} y' + \frac{e^{2x}}{x^3}\left(2 - \frac{3}{x}\right) y$$

Bruk $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

$$\frac{e^{2x}}{x^3} y = 2e^{2x} + C$$

$$y = 2x^3 + C_1 x^3 e^{-2x}$$

$\underline{x < 0}$: Integrerende faktor: $-\frac{e^{2x}}{x^3}$

$$-\frac{e^{2x}}{x^3} y' - \left(2 - \frac{3}{x}\right) \frac{e^{2x}}{x^3} = -4e^{2x}$$

Dette er samme likning som for $x > 0$

$$y(x) = 2x^3 + C_2 x^3 e^{-2x}$$

Dermed har vi:

$$y(x) = \begin{cases} 2x^3 + C_1 x^3 e^{-2x} & x > 0 \\ 2x^3 + C_2 x^3 e^{-2x} & x < 0 \end{cases}$$

Observer at selv om diff likningen på formen (*) ikke var definert for $x = 0$, $y(x) = y'(x) = 0$ for alle C_1 og $C_2 \in \mathbb{R}$.

Sjekker for $x > 0$.

$$y'(x) = \left[x^3\left(2 + C_1 e^{-2x}\right)\right]'$$

$$= 3x^2\left(2 + C_1 e^{-2x}\right) - x^3 2 C_1 e^{-2x}$$

$\Rightarrow y'(x) = 0$ (uavh. av C_1 og C_2.

12.5 Finn den generelle løsningen.

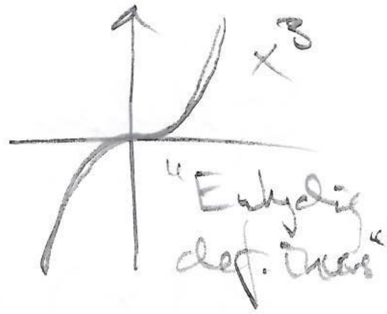
a) $y^2 y' = 5x$ (Separabel, allerede separert!)

$$\int y^2(x) \underbrace{y'(x)dx}_{dy} = \int 5x \, dx$$

$$\int y^2 dy = \frac{5}{2}x^2 + C$$

$$\frac{1}{3}y^3(x) = \frac{5}{2}x^2 + C$$

$$y^3(x) = \underbrace{\frac{15}{2}x^2 + C}_{\text{kan være negativ}}$$



$$y(x) = \sqrt[3]{\frac{15}{2}x^2 + C}$$

b) $\frac{1}{x}y' = \cos x \qquad x \neq 0.$

$$y' = x\cos x$$

$$y = \int x\cos x \, dx + C$$

Løser integralet: Delvis int:

$$\int \underbrace{x}_{u}\,\underbrace{\cos x}_{v'}\, dx = x\sin x - \int \sin x \, dx$$

$$= x\sin x + \cos x$$

$$y(x) = x\sin x + \cos x + C$$

c) $e^y y' = e^{2x}$

$$\int e^y y' \, dx = \int e^{2x} \, dx + C$$

$$e^y = \frac{1}{2} e^{2x} + C$$

$$\underline{\underline{y = \ln\left(\frac{1}{2} e^{2x} + C\right)}}$$

B3.11 Finn en løsning av differensial-likningen

$$x y' + 3y = 3x^{-3/2} \quad (x > 0)$$

$$y' + \frac{3}{x} y = 3x^{-5/2}$$

$$e^{3\ln x} = x^3$$ ← Integrerende faktor

$$x^3 y' + 3x^2 y = 3x^{1/2}$$

$$x^3 y = \int 3x^{1/2}\,dx + C$$

$$x^3 y = 3 \cdot \frac{2}{3} x^{3/2} + C$$

$$y = 2x^{-\frac{3}{2}} + Cx^{-3}$$

Sjekk: $$y' = -3x^{-\frac{5}{2}} - 3Cx^{-4}$$

$$xy' + 3y = \left(-3x^{-\frac{3}{2}} \cancel{-3Cx^{-3}}\right)$$

$$+3\left(2x^{-\frac{3}{2}} \cancel{- Cx^{-3}}\right)$$

$$= 3x^{-\frac{3}{2}}$$

ok.

B 3.12

a) Løs differensiallikningen

$$xy' + 2y = \cos x \quad , \quad x > 0.$$

$$y' + \frac{2}{x} y = \frac{1}{x} \cos x$$

Finner integrerende faktor:

$$\int \frac{2}{x} dx = 2 \ln x + C$$

$$C = 0. \quad e^{2 \ln x} = x^2.$$

$$x^2 y' + 2xy = x \cos x$$

$$x^2 y = \int x \cos x \, dx + C$$

$$\int \underset{u}{x} \underset{v'}{\cos x} \, dx = x \sin x - \int \sin x \, dx$$

$$= x \sin x + \cos x + C$$

$$\underline{\underline{y(x) = \frac{1}{x} \sin x + \frac{1}{x^2} \cos x + \frac{C}{x^2}}}$$

b) Løs initialverdiproblemet

$$y'' - 2y' + 5y = 0 \ , \ y(0) = 3, \ y'(0) = -1.$$

Kar. likn: $r^2 - 2r + 5 = 0$

$$r = \frac{2 \pm \sqrt{4 - 4 \cdot 1 \cdot 5}}{2} = 1 \pm 4i$$

Løsningsmetode s. 289 gir

$$\underline{y(x) = e^x \left(C\cos(4x) + D\sin(4x) \right)}$$

$y(0) = 3 \Rightarrow C = 3$

$y'(0) = -1 \Rightarrow$

$$y'(x) = e^x \left(C\cos(4x) + D\sin(4x) \right) + e^x \left(C(-4\sin(4x)) + D(4\cos(4x)) \right)$$

$y'(0) = C + 4D$

$$C + 4D = -1 \overset{(C=3)}{\Rightarrow} 4D = -4$$

$$\underline{D = -1}$$