Plenum uke 46

Oppgaver: A12: 6, 9, 10, 12
B8: 1, 3.

12.6 Finn den generelle løsningen til difflikningen.

a) $y'' + 2y' - 5y = 0$

Kar. likn: $r^2 + 2r - 5 = 0$

$$r = \frac{-2 \pm \sqrt{4 - 4 \cdot 1 \cdot (-5)}}{2} = -1 \pm \sqrt{1+5}$$

$$= -1 \pm \sqrt{6}.$$

$r_1 = -1+\sqrt{6}$, $r_2 = -1-\sqrt{6}$.

Løsningen har dermed formen:

$$y(x) = Ce^{(-1+\sqrt{6})x} + De^{-(1+\sqrt{6})x}$$

b) $y'' - \frac{1}{2}y' - 2y = 0$

Kar. likn: $r^2 - \frac{1}{2}r - 2 = 0$

$$r = \frac{\frac{1}{2} \pm \sqrt{\frac{1}{4} + 4 \cdot 1 \cdot 2}}{2} = \frac{1}{4} \pm \frac{1}{2}\sqrt{\frac{1}{4} + \frac{8 \cdot 4}{4}}$$

$$= \frac{1}{4} \pm \frac{1}{4}\sqrt{33}$$

Dette gir oss løsningene:

$$\underline{y(x) = Ce^{\frac{1}{4}(1+\sqrt{33})x} + De^{\frac{1}{4}(1-\sqrt{33})x}}$$

c) $y'' + 4y' + 4y = 0$

Kar. likn: $r^2 + 4r + 4 = 0$

$$r = \frac{-4 \pm \sqrt{16 - 4 \cdot 1 \cdot 4}}{2} = \underline{\underline{-2}}$$

Tilfelle 2: Én reell rot. Løsningene er

$$\underline{\underline{y(x) = (C + Dx)e^{-2x}}}$$

12.9 Løs initialverdiproblemene.

a) $y' - y = 4$, $y(0) = 2$

[$y' + f(x)y = g(x) \Rightarrow$ Int. faktor $e^{F(x)}$, $F'(x) = f$]

$f(x) = -1$. $F(x) = -x$

$$e^{-x}y' - e^{-x}y = 4e^{-x}$$

$$(e^{-x}y)' = 4e^{-x}$$

$$e^{-x}y = -4e^{-x} + C \quad | \, e^{x}$$

$$y(x) = Ce^{x} - 4$$

(Kan deles inn i generell løsning av homogen og partikulærløsning)

$y(0) = 2 \Rightarrow C - 4 = 2$, $C = 6$

$$\underline{y(x) = 6e^{x} - 4}$$

b) $y' + xy = x$, $y(0) = 0$.

$F(x) = \frac{1}{2}x^2$

$$e^{\frac{1}{2}x^2} y' + x e^{\frac{1}{2}x^2} y = x e^{\frac{1}{2}x^2}$$

$$e^{\frac{1}{2}x^2} y = \int x e^{\frac{1}{2}x^2} dx$$

$$e^{\frac{1}{2}x^2} y(x) = e^{\frac{1}{2}x^2} + C$$

$$y(x) = 1 + Ce^{-\frac{1}{2}x^2}$$

$y(0) = 0 \Rightarrow 1 + C = 0 \Rightarrow C = -1$

$$\underline{y(x) = 1 - e^{-\frac{1}{2}x^2}}$$

c) $y' + 2y = x^2$, $y(1) = 2$.

$$e^{2x} y' + 2e^{2x} y = x^2 e^{2x}$$

$$e^{2x} y = \int x^2 e^{2x} dx$$

Vi må løse integralet. Bruker delvis int:

$$\int x^2 e^{2x}\,dx \qquad u'(x) = e^{2x}, \quad v(x) = x^2,$$
$$u(x) = \frac{1}{2}e^{2x}, \quad v'(x) = 2x$$

$$\int x^2 e^{2x}\,dx = \frac{1}{2}x^2 e^{2x} - \int x e^{2x}\,dx$$

$$u'(x) = e^{2x} \qquad v(x) = x$$
$$u(x) = \frac{1}{2}e^{2x} \qquad v'(x) = 1$$

$$\int x e^{2x}\,dx = \frac{1}{2}x e^{2x} - \frac{1}{2}\int e^{2x}\,dx$$
$$= \frac{1}{2}x e^{2x} - \frac{1}{4}e^{2x} + C$$

Dermed har vi

$$\int x^2 e^{2x}\,dx = \frac{1}{2}x^2 e^{2x} - \frac{1}{2}x e^{2x} + \frac{1}{4}e^{2x} + C$$
$$= \left(\frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4}\right)e^{2x} + C$$

$$\underline{y(x) = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4} + Ce^{-2x}}$$

$y(1) = 2 \Rightarrow \frac{1}{4} + Ce^{-2} = 2$

$y(1) = \frac{1}{4} + Ce^{-2}$

$Ce^{-2} = \frac{7}{4}$

$\underline{C = \frac{7}{4}e^{2}}$

$$\underline{\underline{y(x) = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4} + \frac{7}{4}e^{2(1-x)}}}$$

12.10 Løs initialverdiproblemene.

a) $y' + \frac{x}{2}y = 2x \quad , y(2) = 5$

$F(x) = \frac{1}{4}x^2$

$$e^{\frac{x^2}{4}}y' + \frac{x}{2}e^{\frac{x^2}{4}}y = 2xe^{\frac{x^2}{4}}$$

$$e^{\frac{x^2}{4}}y = \underbrace{\int 2xe^{\frac{x^2}{4}}\,dx}_{I}$$

Bruker substitusjon:

$u(x) = x^2 \qquad \frac{du}{dx} = 2x$

$$I = \int 2x e^{\frac{x^2}{4}} dx = \int e^{u/4} du = 4e^{\frac{u}{4}} + C$$

$$= 4e^{\frac{x^2}{4}} + C$$

$$y(x) = 4 + Ce^{-\frac{x^2}{4}}$$

$$y(2) = 5 \Rightarrow \quad 4 + Ce^{-1} = 5$$

$$Ce^{-1} = 1$$

$$\underline{C = e}$$

$$\underline{\underline{y(x) = 4 + e^{1-\frac{x^2}{4}}}}$$

b) $y\cos(x) + y' = \cos(x)$

$$\left(y' + \cos x \, y = \cos x \right)$$

$F(x) = \sin x$

$$\left(e^{\sin x} y\right)' = \cos x \, e^{\sin x}$$

$$e^{\sin x} y = \int \cos x \, e^{\sin x} dx$$

$$= e^{\sin x} + C$$

$$y(x) = 1 + Ce^{-\sin x}$$

$y(0) = \pi \Rightarrow 1 + C = \pi$, $\underline{C = \pi - 1}$

$$\underline{\underline{y(x) = 1 + (\pi - 1)e^{-\sin x}}}$$

c) $\frac{y'}{y} - 2x - 1 = 0$, $y(1) = e$

($y' - (2x+1)y = 0$, eller separabel)

$$F(x) = -(x^2 + x)$$

$$\left(e^{-x^2 - x} y\right)' = 0$$

$$y(x) = Ce^{x^2 + x}$$

$y(1) = e \Rightarrow Ce^2 = e \Rightarrow C = e^{-1}$

$$y(x) = e^{x^2+x-1}$$

d) $y(\underbrace{\sin^2 x - 1}_{-\cos^2 x}) = y'$, $y(0) = 2$.

$$y' + \cos^2 x\, y = 0$$

$$\int \cos^2 x\, dx = \int \cos x \cos x\, dx$$

$u'(x) = \cos x \qquad v = \cos x$

$u(x) = \sin x \qquad v' = -\sin x$

$$\int \cos^2 x\, dx = \cos x \sin x + \int \underbrace{\sin^2 x}_{1-\cos^2 x}\, dx$$

$$= \cos x \sin x + x - \int \cos^2 x\, dx$$

$$\int \cos^2 x\, dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$$

$$F(x) = \frac{1}{2}\cos x \sin x + \frac{1}{2}x \qquad (C=0)$$

$$\left(e^{F(x)}y\right) = C$$

$$y(x) = Ce^{-F(x)}$$

$$= C\exp\left(-\frac{1}{2}\cos x\sin x - \frac{1}{2}x\right)$$

$$y(0) = 2 \implies C = 2$$

$$\underline{y(x) = 2\exp\left(-\frac{1}{2}\cos x\sin x - \frac{1}{2}x\right)}$$

12.12 $L(t)$: Antall laks i en innsjø.
(t måles i uker)

$$L'(t) = -k\sqrt{L(t)} \qquad k > 0.$$

"Hvor lang tid tar det før all laksen i innsjøen er død hvis vi starter med 1000 laks og har 720 igjen etter 4 uker?"

$$L(0) = 1000 \quad , \quad L(4) = 720$$

Finner generell løsning.

$$L'(t) = -k\sqrt{L(t)} \qquad L(t) \geq 0$$

$$\frac{1}{\sqrt{L(t)}} L'(t) = -k \qquad | \text{ integrerer i } t.$$

$$\underbrace{\int \frac{1}{\sqrt{L(t)}}\, dL} = -kt + C$$

$$\int \frac{1}{\sqrt{L}}\, dL = 2 \int \frac{1}{2\sqrt{L}}\, dL = 2\sqrt{L} + C$$

$$2\sqrt{L(t)} = -kt + C$$

$$L(t) = \left(-\frac{1}{2}kt + C\right)^2$$

Må bestemme k og C.

$$L(0) = 1000 \Rightarrow C^2 = 1000 \Leftrightarrow C = \sqrt{1000} = 10\sqrt{10}$$

$$L(4) = 720 \Rightarrow (-2k + C)^2 = 720$$

$C - 2k = \pm\sqrt{720} = \pm\sqrt{4^2 \cdot 3^2 \cdot 5}$

$= \pm 12\sqrt{5}$

720	2	
360	2	2^4
180	2	
90	2	
45	3	3^2
15	3	
5	5	

$C - 2k = \pm 12\sqrt{5}$

$2k = C \pm 12\sqrt{5}$

$k = \frac{C}{2} \pm 6\sqrt{5}$

$k_1 \approx 2.4$ eller $\underset{k_2}{29.2}$

Vi skal finne tidspunktet t_0 for $L(t_0) = 0$

$C - \frac{1}{2} k t_0 = 0$

$t_0 = \frac{2}{k} C$

k_1 gir $\underline{\underline{t_0 \approx 26.4}}$

k_2 gir $t_0 \approx 2.2$

Siden det skal være 720 laks igjen etter 4 uker må $k = k_1$.

B3.1 $y(t)$: Mengde thorium-234 ved tiden t.

$$\frac{dy}{dt} = -ay \qquad (a>0)$$

a) Løs differensiallikningen (B.1) og finn halveringstiden når "100 mg av stoffet etter 7 dager er redusert til 82.04 mg".

$y(0) = 100$, $y(7) = 82.04$.

(t målt i dager / sekunder?)

$$y' + ay = 0 \qquad \Big| \, e^{at}$$

$$(e^{at} y)' = 0$$

$$e^{at} y = C \Rightarrow \underline{y(t) = Ce^{-at}}$$

$y(0) = 100 \Rightarrow C = 100$

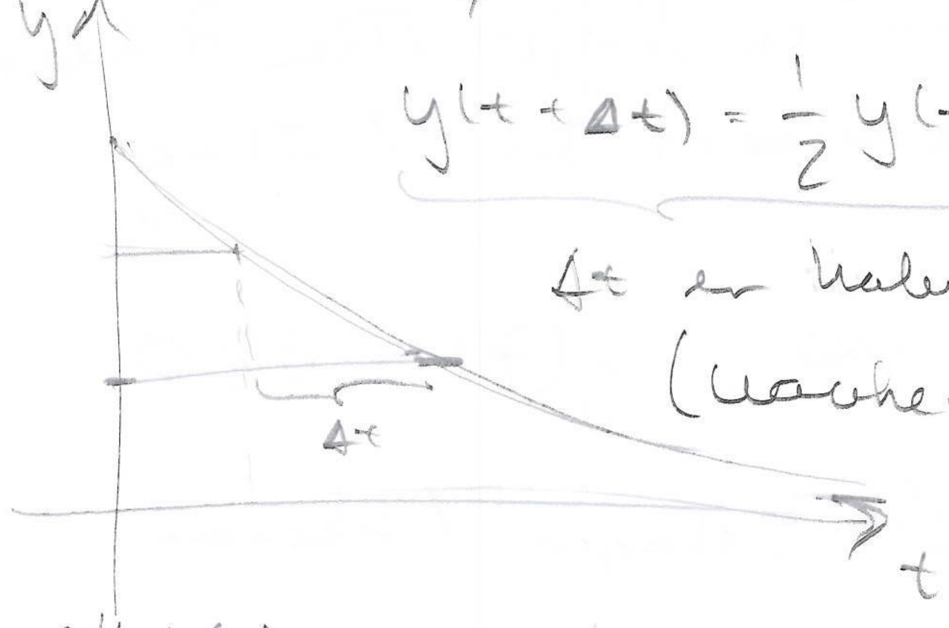
Bestemmer a.

$y(7) = 82.04 \Rightarrow 100e^{-a7} = 82.04$

$$e^{-a7} = 0.8204$$

$$-a7 = \ln(0.8204)$$

$$a = -\frac{\ln(0.8204)}{7} \approx \underline{\underline{0.028}}$$



$$\underbrace{y(t+\Delta t) = \frac{1}{2}y(t)}$$

Δt er halveringstiden.
(uavhengig av t!)
(og C)

$$\cancel{C}e^{-a(t+\Delta t)} = \frac{1}{2}\cancel{C}e^{-at}$$

$$\cancel{e^{-at}}e^{-a\Delta t} = \frac{1}{2}\cancel{e^{-at}}$$

$$e^{-a\Delta t} = \frac{1}{2} \qquad -a\Delta t = \ln\left(\frac{1}{2}\right)$$

$$\Delta t = -\ln\left(\frac{1}{2}\right)/a$$

$$\Delta t = \frac{\ln 2}{a} \approx \underline{\underline{24.8}}$$

Halveringstiden for thorium-234 er ca. 24.8 dager.

b) Anta at det tilføres 1 mg av isotopen per dag. Beregn y(t) og finn grenseverdien y(t) går mot når $t \to \infty$.

$$y'(t) = -ay + 1 \qquad \left(y' \text{ har enhet mg/dag}\right)$$

$$y' + ay = 1 \qquad \Big| \, e^{at}$$

$$\underbrace{e^{at}y' + ae^{at}y} = e^{at}$$

$$(e^{at}y)' = e^{at}$$

$$e^{at}y = \int e^{at}\,dt + C$$

$$y(t) = e^{-at}\left(\frac{1}{a}e^{at}\right) + Ce^{-at}$$

$$= \underline{\underline{\frac{1}{a} + Ce^{-at}}}$$

Bestemmer C: $y(0) = 100$

$$\frac{1}{a} + C = 100 \Rightarrow C = 100 - \frac{1}{a}$$

$$\simeq 64.3$$

$$\underline{\underline{y(t) \simeq 35.7 + 64.3e^{-0.028t}}}$$

$$\lim_{t \to \infty} y(t) \simeq \underline{\underline{35.7}}$$

B 3.3

$X(t)$: Antall friske personer ved tiden t (målt i dager)

$y(t)$: Antall smittede ved tiden t.

$x(t) + y(t) = N+1$ (konserveringsuttr.)

Anta $\boxed{\dfrac{dy}{dt} = \beta x y}$, $\beta > 0$ konst. "spesifikk smitterate"

a) Løs differensiallikningen.

Husk $x(t) = N+1 - y(t)$

$y' = \beta(N+1-y)y$ Anta $y(t) > 0$

$$\frac{y'}{(N+1-y)y} = \beta \quad \text{Integrerer it.}$$

$$\int \frac{1}{(N+1-y)y}\,dy = \beta t + C$$

For å løse integralet bruker vi delbrøksoppspalting.

$$\frac{1}{(N+1-y)y} = \frac{A}{y} + \frac{B}{N+1-y}$$

$$1 = A(N+1-y) + By \quad (\forall y)$$

$$y = N+1 \implies 1 = B(N+1)$$

$$y = 0 \implies 1 = A(N+1)$$

$$A = B = \frac{1}{N+1}$$

$$\frac{1}{(N+1-y)y} = \frac{1}{N+1}\left(\frac{1}{y} + \frac{1}{(N+1)-y}\right)$$

(For hvilke y er dette definert?)

$$\int \frac{1}{(N+1-y)y}\,dy = \frac{1}{N+1}\left(\ln(y) - \ln(N+1-y)\right)$$

Legg merke til at uttrykket er definert for $y \in (0, N+1)$

$$\frac{1}{N+1} \ln\left(\frac{y}{N+1-y}\right) = \beta t + C$$

Ny konst.

$$\frac{y}{N+1-y} = Ce^{(N+1)\beta t}$$

$$y = (N+1-y)Ce^{(N+1)\beta t}$$

$$y = (N+1)Ce^{(N+1)\beta t} - yCe^{(N+1)\beta t}$$

$$y + yCe^{(N+1)\beta t} = (N+1)Ce^{(N+1)\beta t}$$

$$y(t) = \frac{(N+1)Ce^{(N+1)\beta t}}{1+Ce^{(N+1)\beta t}}$$

$$= (N+1)\frac{Ce^{(N+1)\beta t}}{1+Ce^{(N+1)\beta t}}$$

b) Anta $N = 100\,000$, $\beta = 0.00000013$

Hvor lang tid tar det før 90% av befolkningen er smittet?

Må bestemme C. $y(0) = 1$.

(Dette kunne vi gjort i a)

$$y(0) = 1 \Rightarrow (N+1)\frac{C}{1+C} = 1$$

$$\frac{1}{N+1} = \frac{C}{1+C}, \quad \frac{1}{N+1}(1+C) = C$$

$$\frac{1}{N+1} + C\left(\frac{1}{N+1}\right) = C$$

$$\frac{1}{N+1} = C\left(1 - \frac{1}{N+1}\right) \quad \Big|\, N+1$$

$$1 = \underbrace{C\,(N+1-1)}_{=\,CN}$$

$$\underline{\underline{C = \frac{1}{N}}}$$

$C = \frac{1}{N}$ gir oss

$$y(t) = (N+1)\,\frac{e^{(N+1)\beta t}}{N + e^{(N+1)\beta t}}$$

$$= \underline{\underline{\frac{(N+1)e^{(N+1)\beta t}}{N + e^{(N+1)\beta t}}}}$$

När er 90 % smittet?

$$\boxed{\frac{y(t)}{N+1} = \frac{9}{10}} \Rightarrow \frac{9}{10} = \frac{e^{(N+1)\beta t}}{N + e^{(N+1)\beta t}}$$

$$\frac{9}{10} = \frac{1}{Ne^{-(N+1)\beta t} + 1}$$

$$\frac{9}{10}\left(Ne^{-(N+1)\beta t} + 1\right) = 1$$

$$Ne^{-(N+1)\beta t} = \frac{10}{9} - 1 = \frac{1}{9}$$

$$e^{-(N+1)\beta t} = \frac{1}{9N}$$

$$-(N+1)\beta t = \ln\left(\frac{1}{9N}\right) = -\ln(9N)$$

$$t = \frac{\ln(9N)}{(N+1)\beta} \approx \underline{\underline{105.5}}$$

90 % av befolkningen er smittet etter 105.5 dager.