HØST 2011

① (inhomogent)

$$\begin{cases} x+y+z=0 \\ 2x-y+2z=3 \\ 2x+y+\alpha z=1 \end{cases}$$

$\alpha \in \mathbb{R}$ / parameter

Vendelig mange løsninger for en α?

$\det A = 0$ (uendelig mange / ingen løsn.) $\quad A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ 2 & 1 & \alpha \end{bmatrix}$

$\det A \neq 0 \Rightarrow$ EN LØSNING

$$0 = \det A = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ 2 & 1 & \alpha \end{vmatrix}$$

$$= 1 \cdot \begin{vmatrix} -1 & 2 \\ 1 & \alpha \end{vmatrix} - 1 \cdot \begin{vmatrix} 2 & 2 \\ 2 & \alpha \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= -\alpha - 2 - [2\cdot\alpha - 2\cdot 2] + [2 - 2\cdot(-1)]$$

$$= -\alpha - 2 - 2\alpha + 4 + 2 + 2$$

$$= -3\alpha + 6$$

$3\alpha = 6$

$\alpha = 2$ Kan gi ingen eller uendelig mange løsn.

Vi setter inn:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & -1 & 2 & 3 \\ 2 & 1 & \boxed{2} & 1 \end{array}\right] \underset{-2R_1 \text{ til } R_3}{\overset{-2R_1 \text{ til } R_2}{\sim}} \left[\begin{array}{cccc} 1 & 1 & 1 & 0 \\ 0 & -3 & 0 & 3 \\ 0 & -1 & 0 & 1 \end{array}\right] \underset{-\frac{1}{3}R_2 \text{ til } R_3}{\overset{-\frac{1}{3}\cdot R_2}{\sim}} \left[\begin{array}{cccc} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\overset{-R_2 \text{ til } R_1}{\sim} \left[\begin{array}{cccc} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Fri

$z = s, \quad s \in \mathbb{R}$

$x + z = 1 \quad x = 1 - s$

$y = -1$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \left\{ \begin{bmatrix} 1-s \\ -1 \\ s \end{bmatrix}, s \in \mathbb{R} \right\}$$ er løsningsmengden (linje i $\mathbb{R}^3$)

$$= \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} s + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, s \in \mathbb{R} \right\}$$

2011 4

$$y'' + 2y' + 2y = 0 \qquad y(0) = 1$$

$$\boxed{y'(0) = 1}$$

$$r^2 + 2r + 2 = 0$$

Bruk abc!!

$$\underline{r = -1 \pm i} \qquad a + ib \qquad a = -1, \; b = 1$$

$$y(x) = e^{ax}\left[C\cos(bx) + D\sin(bx)\right]$$

$$\underline{y(x) = e^{-x}\left[C\cos(x) + D\sin(x)\right]}, \quad C, D \in \mathbb{R}$$

$$\underline{1} = y(0) = \underbrace{e^0}_{=1}\left[C\underbrace{\cos 0}_{=1} + D\underbrace{\sin 0}_{=0}\right] = \underline{C}$$

$$y(x) = \underbrace{e^{-x}}\underbrace{\left[\cos(x) + D\sin(x)\right]}$$

$$y'(x) = e^{-x}\cdot(-1)\left[\cos(x) + D\sin(x)\right] + e^{-x}\cdot\left[-\sin(x) + D\cos(x)\right]$$

$$1 = y'(0) = -\underbrace{e^0}_{=1}\left[\underbrace{\cos(0)}_{=1} + \underline{D\underbrace{\sin(0)}_{=0}}\right] + \underbrace{e^0}_{=1}\left[-\underbrace{\sin 0}_{=0} + D\underbrace{\cos(0)}_{=1}\right]$$

$$= -1 + D$$

$$\underline{D = 2}$$

$$y(x) = \underline{\underline{e^{-x}\left[\cos(x) + 2\sin(x)\right]}}$$

V 2010 ④ $P > 0 \quad P \neq 1$

a) $M = \begin{bmatrix} 1 & 1-P \\ 0 & P \end{bmatrix}$ Egenverdier og egenvektorer

$M\mathbf{v} = \lambda \mathbf{v} \qquad (\lambda, \mathbf{v})$

$M\mathbf{v} - \lambda\mathbf{v} = \boxed{(M-\lambda I)\mathbf{v} = \mathbf{0}}$

- $\det(M-\lambda I) = 0$

Finner egenverdier

$$0 = \begin{vmatrix} 1-\lambda & 1-P \\ 0 & P-\lambda \end{vmatrix} = (1-\lambda)(P-\lambda)$$

$\lambda_1 = 1$

$\lambda_2 = P$

Finne egenvektorer:

$\lambda_1 = 1$: $\mathbf{v}_1 = \begin{bmatrix} p_1 \\ q_1 \end{bmatrix}$ $\begin{bmatrix} 1-1 & 1-P & 0 \\ 0 & P-1 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1-P & 0 \\ 0 & P-1 & 0 \end{bmatrix}$

$\frac{1}{1-P} \cdot R_1$, $\frac{1}{P-1} \cdot R_2$ $\sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \overset{-R_1 \text{ til } R_2}{\sim} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Fri

$p_1 = s, \quad s \in \mathbb{R}$

$q_1 = 0$

$\mathbf{v}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot s, \ s \in \mathbb{R} \right\}$ $s = 1$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_2 = P$ $\mathbf{v}_2 = \begin{bmatrix} p_2 \\ q_2 \end{bmatrix}$ $(M-\lambda_2 I)\mathbf{v}_2 = 0$ $\begin{bmatrix} 1-P & 1-P & 0 \\ 0 & P-P & 0 \end{bmatrix} \sim \begin{bmatrix} 1-P & 1-P & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\frac{1}{1-P} \cdot R_1$ $\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Fri

$q_2 = t, \quad t \in \mathbb{R}$

$p_2 + q_2 = 0$

$p_2 = -t$

$\mathbf{v}_2 = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot t, \ t \in \mathbb{R} \right\}$ $t = 1$ $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

b) $\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1-P \\ 0 & P \end{bmatrix} \cdot \begin{bmatrix} x_n \\ y_n \end{bmatrix}$ $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ $\begin{bmatrix} x_n \\ y_n \end{bmatrix} = ?$

$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = M \cdot \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = M \cdot \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = M \cdot M \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = M^2 \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

$\vdots$

$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = M^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ $M \cdot \mathbf{v} = \lambda \mathbf{v}$ $M^n \mathbf{v} = \lambda^n \mathbf{v}$

- $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = a \cdot \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\mathbf{v}_1} + b \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\mathbf{v}_2} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ — $\begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 1 \end{bmatrix} \overset{R_2 \text{ til } R_1}{\sim} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \end{bmatrix}$ $a = 4$, $b = 1$

$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = M^n \left(4 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$

$= 4 \cdot M^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} + M^n \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$= 4 \cdot 1^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} + P^n \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 4 - P^n \\ P^n \end{bmatrix}$

$P > 0$

$P \neq 1$

$\lim_{n\to\infty} P^n = 0$ når $P < 1$

$= \infty$ $P > 1$

c) $\lim_{n\to\infty} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$

$= \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

Grenseverdien eksisterer når $P < 1$.

H 2011 (2)

$y = f(x) \quad y' = f'(x) = x \cdot \ln x \qquad f(1) = 0$

$\int \underbrace{y' d\cancel{x}}_{dy} = \int \overset{v'}{x} \overset{u}{\ln x}\, dx$

$y = \frac{1}{2}x^2 \cdot \ln x - \int \frac{1}{x} \cdot \frac{1}{2} x^2 dx + C$

DELVIS

$u = \ln x \qquad u' = \frac{1}{x}$

$v' = x \qquad v = \frac{1}{2}x^2$

$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x^{-1} \cdot x^2 dx + C$

$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x \, dx + C$

$= \frac{1}{2}x^2 \ln x - \frac{1}{2} \cdot \frac{1}{2}x^2 + C$

$e^{\ln 1} = 1$

$y(x) = f(x) = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C \quad , C \in \mathbb{R}$

$f(1) = 0 = \frac{1}{2} \cdot 1^2 \cdot \underbrace{\ln 1}_{0} - \frac{1}{4} \cdot 1^2 + C$

$0 = -\frac{1}{4} + C$

$C = \frac{1}{4}$

$y(x) = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + \frac{1}{4}$

V 2010 1 a)

$$f(x) = 3xe^{x^2}$$

$$\int f(x)\,dx = \int 3xe^{x^2}\,dx$$

$$= 3\int e^u \cdot \frac{1}{2}\,du$$

$$= \frac{3}{2}e^u + C, \quad C \in \mathbb{R}$$

$$= \frac{3}{2}e^{x^2} + C, \quad C \in \mathbb{R}$$

SUBST.

$u = x^2$

$\frac{du}{dx} = 2x \quad \Big|\ \text{"}\cdot dx\text{"}, \ \cdot\frac{1}{2}$

$\frac{1}{2}du = x\,dx$

b)

$$y' + xy = xe^{\frac{1}{2}x^2} \qquad y(0) = 1$$

Int. fakt

$f(x) = x$

$F(x) = \frac{1}{2}x^2$

$\boxed{\cdot\ e^{F(x)} = e^{\frac{1}{2}x^2}}$

$e^{-F(x)} = e^{-\frac{1}{2}x^2}$

$$y(x) = e^{-\frac{1}{2}x^2}\int e^{\frac{1}{2}x^2} \cdot xe^{\frac{1}{2}x^2}\,dx + \boxed{Ce^{-\frac{1}{2}x^2}}, \quad C \in \mathbb{R}$$

$$e^{-F(x)}\int e^{F(x)} \cdot g(x)\,dx + Ce^{-F(x)}$$

$$= e^{-\frac{1}{2}x^2}\int xe^{\frac{1}{2}x^2+\frac{1}{2}x^2}\,dx + Ce^{-\frac{1}{2}x^2}$$

$$= e^{-\frac{1}{2}x^2}\int x \cdot e^{x^2}\,dx + Ce^{-\frac{1}{2}x^2}$$

a) $\int 3xe^{x^2}\,dx$

svar fra a) $\cdot\frac{1}{3}$

sette $C=0$ i a)

$$= e^{-\frac{1}{2}x^2} \cdot \left[\frac{1}{\cancel{3}} \cdot \frac{\cancel{3}}{2} \cdot e^{x^2}\right] + Ce^{-\frac{1}{2}x^2}$$

$$a^m \cdot a^n = a^{m+n}$$

$$= \frac{1}{2}e^{-\frac{1}{2}x^2 + x^2} + Ce^{-\frac{1}{2}x^2}$$

$$= \frac{1}{2}e^{\frac{1}{2}x^2} + Ce^{-\frac{1}{2}x^2}, \quad C \in \mathbb{R}$$

$$y(0) = 1 = \frac{1}{2}e^0 + Ce^0 = \frac{1}{2} + C \qquad C = \frac{1}{2}$$

$$y(x) = \frac{1}{2}e^{\frac{1}{2}x^2} + \frac{1}{2}e^{-\frac{1}{2}x^2}$$