A12 1,2,3

$$y' + f(x)y = g(x)$$

$F(x) = \int f(x)\,dx$ F én antiderivert til f.

$$e^{F(x)}y' + e^{F(x)}f(x)y = e^{F(x)}g(x)$$

$$\left(e^{F(x)}y\right)' = e^{F(x)}g(x)$$

$$e^{F(x)}y = \int e^{F(x)}g(x)\,dx + C$$

$$y = e^{-F(x)}\int e^{F(x)}g(x)\,dx + e^{-F(x)}C$$

A12

1 a) $y' + \frac{1}{x} y = \sin x$

$f(x) = \frac{1}{x}, \quad g(x) = \sin x, \quad x > 0$

$F(x) = \int \frac{1}{x} dx = \ln|x| = \ln x$

$e^{F(x)} = e^{\ln x} = x$

$\int e^{F(x)} g(x) dx = \int x \sin x dx$

$v' = \sin x, \quad u = x$

$v = -\cos x \quad u' = 1$

$\int x \sin x dx = -x\cos x - \int (-\cos x) dx$

$= -x\cos x + \int \cos x dx = -x\cos x + \sin x + C$

$e^{-F(x)} = e^{-\ln x} = \frac{1}{x}$

$y = \frac{1}{x}\left(-x\cos x + \sin x\right] + \frac{C}{x}$

$= -\cos x + \frac{\sin x + C}{x}$

b)

$$\frac{y'}{4x} + y = 2 \qquad x \neq 0$$

$$y' + 4xy = 8x \ , \ f(x) = 4x, \ g(x) = 8x$$

$$F(x) = \int 4x\,dx = 2x^2, \quad e^{F(x)} = e^{2x^2}$$

$$\int e^{F(x)} g(x)\,dx = \int 8x e^{2x^2} dx = \int 2 e^u \, du$$

$$u = 2x^2$$
$$du = 4x\,dx \qquad = 2e^u + C$$
$$8x\,dx = 2\,du \qquad = 2e^{2x^2} + C$$

$$y = e^{-2x^2}(2e^{2x^2}) + Ce^{-2x^2} = 2 + Ce^{-2x^2}$$

c) $2y' + y = e^x, \quad y' + \frac{1}{2}y = \frac{1}{2}e^x, \ f(x) = \frac{1}{2}$

$$g(x) = \frac{1}{2}e^x$$

$$F(x) = \int \frac{1}{2}dx = \frac{x}{2}, \quad e^{F(x)} = e^{\frac{x}{2}}$$

$$e^{-F(x)} = e^{-\frac{x}{2}}, \quad \int e^{F(x)} g(x)\,dx = \int e^{\frac{x}{2}} \frac{1}{2} e^x dx =$$

$$= \frac{1}{2}\int e^{\frac{3}{2}x} dx = \frac{1}{2} \cdot \frac{2}{3} e^{\frac{3}{2}x} + C =$$

$$= \frac{1}{3} e^{\frac{3}{2}x} + C, \quad y = e^{-\frac{x}{2}}\left(\frac{1}{3}e^{\frac{3}{2}x}\right) + Ce^{-\frac{x}{2}}$$

$$= \underline{\frac{1}{3}e^x + Ce^{-\frac{x}{2}}}$$

12.2

a) $y' + 2xy = x \quad , \quad f(x) = 2x, \; g(x) = x$

$F(x) = \int 2x\,dx = x^2, \quad \int e^{F(x)} g(x)\,dx =$

$= \int e^{x^2} x\,dx = \int \frac{1}{2} e^u\,du = \frac{1}{2} e^u + C = \frac{1}{2} e^{x^2} + C$

$u = x^2$
$du = 2x\,dx$
$x\,dx = \frac{1}{2}du \quad ; \quad y = e^{-F(x)} \int e^{F(x)} g(x)\,dx + e^{-F(x)} C$

$= e^{-x^2}\left(\frac{1}{2} e^{x^2}\right) + e^{-x^2} C = \underline{\frac{1}{2} + C e^{-x^2}}$

b) $y' + y = e^x, \quad f(x) = 1 \;, \quad g(x) = e^x$

$F(x) = \int f(x)\,dx = \int dx = x, \quad \int e^{F(x)} g(x)\,dx$

$= \int e^x \cdot e^x\,dx = \int e^{2x}\,dx = \frac{1}{2} e^{2x} + C$

$y = e^{-x}\left(\frac{1}{2} e^{2x}\right) + C e^{-x} = \frac{1}{2} e^x + C e^{-x}$

c) $y' + 2y = xe^x$, $f(x) = 2$, $g(x) = xe^x$

$F(x) = \int 2\,dx = 2x$,

$\int e^{F(x)} g(x)\,dx = \int e^{2x} \cdot x e^x\,dx = \int x e^{3x}\,dx =$

$v' = e^{3x}$ $\quad u = x$

$v = \frac{1}{3} e^{3x}$ $\quad u' = 1$

$= \underline{\frac{x}{3} e^{3x} - \int \frac{1}{3} e^{3x}\,dx} = \underline{\frac{x}{3} e^{3x} - \frac{1}{9} e^{3x} + C}$

$\left(\begin{array}{l} g(x) = 3x, \quad f(u) = e^u, \quad (e^{3x})' = (f(g(x)))' = f'(g(x))\,g'(x) \\ f'(u) = e^u \qquad\qquad\qquad\qquad = e^{3x} \cdot 3 \end{array} \right)$

$y = e^{-F(x)} \left(\int e^{F(x)} g(x)\,dx \right) + C e^{-F(x)}$

$= e^{-2x} \left(\frac{x}{3} e^{3x} - \frac{1}{9} e^{3x} \right) + C e^{-2x} = \underline{\underline{\frac{x}{3} e^x - \frac{1}{9} e^x + C e^{-2x}}}$

A.12

2 d) $(x^2+1)y' + 3xy = 6x$

$$y' + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1}$$

$f(x) = \frac{3x}{x^2+1}$,

$$F(x) = \int \frac{3x}{x^2+1}\,dx = \int \frac{\frac{3}{2}\,du}{u} = \frac{3}{2}\ln|u|$$

$u = x^2+1$

$du = 2x\,dx$

$x\,dx = \frac{1}{2}\,du$

$$= \frac{3}{2}\ln(x^2+1)$$

$g(x) = \frac{6x}{x^2+1}$

$$I = \int e^{F(x)} g(x)\,dx = \int e^{\frac{3}{2}\ln(x^2+1)} \cdot \frac{6x}{x^2+1}\,dx$$

$$e^{\frac{3}{2}\ln(x^2+1)} = e^{\ln((x^2+1)^{\frac{3}{2}})} = (x^2+1)^{\frac{3}{2}}$$

$b\ln a = \ln(a^b)$, $e^{\ln c} = c$

$$I = \int (x^2+1)^{3/2} \frac{6x}{x^2+1}\,dx = \int (x^2+1)^{1/2}\, 6x\,dx =$$

$u = x^2+1,\ du = 2x\,dx$

$6x\,dx = 3\,du$

$$= \int 3\sqrt{u}\,du = 3\,\frac{1}{\frac{1}{2}+1}\,u^{\frac{1}{2}+1} + C$$

$$= 3\,\frac{2}{3}\,u^{3/2} + C = 2u^{3/2} + C = 2(x^2+1)^{3/2} + C$$

$$y = e^{-F(x)}\left(\int e^{F(x)} g(x)\,dx\right) + e^{-F(x)}C$$

$$= (x^2+1)^{-3/2}\left(2(x^2+1)^{3/2}\right) + C(x^2+1)^{-3/2}$$

$$= \underline{\underline{2 + C(x^2+1)^{-3/2}}}$$

12.3

$$xy' + (2x-3)y = 4x^4$$

$$y' + \frac{(2x-3)}{x}y = 4x^3$$, her må vi ha $x \neq 0$

$x > 0$, $f(x) = \frac{2x-3}{x} = 2 - \frac{3}{x}$, $g(x) = 4x^3$

$$F(x) = \int f(x)dx = \int \left(2 - \frac{3}{x}\right)dx = 2x - 3\ln|x| = 2x - 3\ln x$$

(siden $x > 0$) $$e^{F(x)} = e^{2x - 3\ln x} = \frac{e^{2x}}{e^{3\ln x}} =$$

$$= \frac{e^{2x}}{e^{\ln x^3}} = \frac{e^{2x}}{x^3}, \quad e^{-F(x)} = \frac{x^3}{e^{2x}}$$

$$\int e^{F(x)} g(x)dx = \int \frac{e^{2x}}{x^3} 4x^3 dx = \int 4e^{2x}dx =$$

$$= 2e^{2x} + C;$$

$$y_1 = e^{-F(x)}\left(\int e^{F(x)} g(x)\,dx\right) + C_1 e^{-F(x)}$$

$$= \frac{x^3}{e^{2x}}\left(2e^{2x}\right) + C_1 \frac{x^3}{e^{2x}} = 2x^3 + C_1 \frac{x^3}{e^{2x}}$$

$x < 0$,

$$F(x) = 2x - 3\ln|x|$$

$$= 2x - 3\ln(-x)$$

$$e^{F(x)} = e^{2x - 3\ln(-x)}$$

$$= \frac{e^{2x}}{e^{3\ln(-x)}} = \frac{e^{2x}}{e^{\ln(-x)^3}} = \frac{e^{2x}}{-x^3}$$

$$e^{-F(x)} = \frac{-x^3}{e^{2x}}$$

$$\int e^{F(x)} g(x) = \int -\frac{e^{2x}}{x^3} \cdot 4x^3 \, dx$$

$$= -\int e^{2x} dx = -2e^{2x} + C_2$$

$$y_2 = (-x^3)(-2) - C_2 \frac{x^3}{e^{2x}}$$

$$y_2 = 2x^3 + C_2' \frac{x^3}{e^{2x}} \quad (C_2' = -C_2)$$

Får altså løsningen

$$y = \begin{cases} 2x^3 + C_1 \frac{x^3}{e^{2x}} = y_1 & \text{for } x > 0 \\ 2x^3 + C_2 \frac{x^3}{e^{2x}} = y_2 & \text{for } x < 0 \end{cases} \qquad (\text{setter } C_2' = C_2)$$

Men Likningen $xy' + (2x-3)y = 4x^4$

er definert for alle x (også når $x = 0$)

og y_1 og y_2 passer inn for alle x

og $y_1(0) = 0 = y_2(0)$

$$y_1'(x) = 6x^2 + \frac{3x^2 C_1}{e^{2x}} + (-2)C_1 \frac{x^3}{e^{2x}}, \quad y_1'(0) = 0 = y_2'(0)$$

Betyr

$$y(x) = \begin{cases} 2x_3 + C_1 \frac{x^3}{e^{2x}}, & x > 0 \\ 0 & x = 0 \\ 2x_3 + C_2 \frac{x^3}{e^{2x}}, & x < 0 \end{cases}$$

der C_1, C_2 kan velges fritt (og være forskjellig)

er en løsning (fordi den blir en deriverbar

funksjon).