A.12 6, 7, 9, 11

B 3, 20, 22

20) $p(t)$ folketall i New-York

t (målt i år) $t=0$, 1/1-1960

$p(0) = 6.0 \cdot 10^6$, $p(t)$ opptegner

$$\frac{dp}{dt} = 0.56p - 4.0 \cdot 10^{-8}p^2 - 16 \cdot 10^5$$
$$= -4.0 \cdot 10^{-8}(N(p))$$

$N(p) = p^2 - 14 \cdot 10^6 p + 4 \cdot 10^{13}$, $\int \frac{dp}{N(p)} = \int -4 \cdot 10^{-8} dt$

$N(p)=0$, gir $p = \frac{14 \cdot 10^6 \pm \sqrt{(14)^2 10^{12} - 160 \cdot 10^{12}}}{2} =$

$$= \frac{14 \cdot 10^6 \pm \sqrt{196 \cdot 10^{12} - 160 \cdot 10^{12}}}{2} = \frac{10^6(14 \pm 6)}{2} = 10^6 \begin{cases} 10 \\ 4 \end{cases}$$

$$\frac{1}{N(p)} = \frac{A}{p - 10 \cdot 10^6} + \frac{B}{p - 4 \cdot 10^6} = \frac{(A+B)p - 10^6(4A + 10B)}{N(p)}$$

$A + B = 0$ $\quad A = -B$

$-10^6(4A + 10B) = 1$ $\quad -10^6 \cdot 6B = 1$, $B = -\frac{1}{6} \cdot 10^{-6}$, $A = \frac{1}{6} \cdot 10^{-6}$

$$\int \frac{dp}{N(p)} = \frac{1}{6} 10^{-6} \int \left(\frac{1}{p - 10 \cdot 10^6} - \frac{1}{p - 4 \cdot 10^6}\right) dp = \frac{10^{-6}}{6} \ln \left|\frac{p - 10 \cdot 10^6}{p - 4 \cdot 10^6}\right|$$

$$\int \frac{dp}{N(p)} = \frac{10^{-6}}{6} \ln \left| \frac{p - 10 \cdot 10^6}{p - 4 \cdot 10^6} \right|$$

$$= \int -4 \cdot 10^{-8} dt = -4 \cdot 10^{-8} t + C$$

$$\ln \left| \frac{p - 10 \cdot 10^6}{p - 4 \cdot 10^6} \right| = -24 \cdot 10^{-2} t + C$$

$$\left| \frac{p - 10 \cdot 10^6}{p - 4 \cdot 10^6} \right| = e^C e^{-24 \cdot 10^{-2} t}$$

$$\frac{p - 4 \cdot 10^6}{p - 10 \cdot 10^6} = \pm e^{-C} e^{24 \cdot 10^{-2} t} = K e^{24 \cdot 10^{-2} t}$$

$$p(0) = 6 \cdot 10^6, \quad \frac{2 \cdot 10^6}{-4 \cdot 10^6} = K, \quad K = -\frac{1}{2}$$

$$\frac{p(t) - 4 \cdot 10^6}{p(t) - 10 \cdot 10^6} = -\frac{1}{2} e^{24 \cdot 10^{-2} t}$$

$$p(t) - 4 \cdot 10^6 = -\frac{1}{2} e^{24 \cdot 10^{-2} t} (p(t) - 10 \cdot 10^6)$$

$$p(t) \left(1 + \frac{1}{2} e^{24 \cdot 10^{-2} t}\right) = 10^6 \left(4 + 5 e^{24 \cdot 10^{-2} t}\right)$$

$$p(t) = \frac{10^6 (4 + 5 e^{24 \cdot 10^{-2} t})}{1 + \frac{1}{2} e^{24 \cdot 10^{-2} t}} = \underline{\underline{\frac{2 \cdot 10^6 (4 + 5 e^{24 \cdot 10^{-2} t})}{2 + e^{24 \cdot 10^{-2} t}}}}$$

b) Hva skjer når $t \to \infty$

$$p(t) = \frac{2\cdot 10^6 \left(4 + 5e^{24\cdot 10^{-2}t}\right)}{2 + e^{24\cdot 10^{-2}t}}$$

$$= 2\cdot 10^6 \, \frac{4\cdot \underbrace{e^{-24\cdot 10^{-2}t}}_{\to 0,\ t\to\infty} + 5}{2\cdot \underbrace{e^{-24\cdot 10^{-2}t}}_{0,\ t\to\infty} + 1} \xrightarrow{t\to\infty} 10\cdot 10^6$$

(10 millioner)

Så befolkningen vil stabilisere seg på 10 millioner.

c) Når er vekstraten $\left(\frac{dp}{dt}\right)$ størst.

$$\frac{dp}{dt} = 0.56p - 4\cdot 10^{-8}p^2 - 16\cdot 10^5$$

$$\frac{d}{dp}\left(0.56p - 4\cdot 10^{-8}p^2 - 16\cdot 10^5\right)$$

$$= 0.56 - 8\cdot 10^{-8}p = 0 \quad , \quad p = \frac{0.56}{8\cdot 10^{-8}} = \underline{7\cdot 10^6}$$

Må finne t slik at $p(t) = 7\cdot 10^6$

$$p(t) = \frac{2\cdot 10^6\left(4+5e^{24\cdot 10^{-2}t}\right)}{2+e^{24\cdot 10^{-2}t}} = 7\cdot 10^6$$

$$8 + 10e^{24\cdot 10^{-2}t} = 7\left(2+e^{24\cdot 10^{-2}t}\right)$$

$$3\cdot e^{24\cdot 10^{-2}t} = 6 \, , \quad e^{24\cdot 10^{-2}t} = 2$$

$$24\cdot 10^{-2}t = \ln 2$$

$$t = \frac{100}{24}\ln 2 \approx 2.88$$

på slutten av 1962 (16 eller 17 oktober).

A.12

$y'' + py' + qy = 0$, $p, q \in \mathbb{R}$

$r^2 + pr + q = 0$

Ser på rötten r_1, r_2

a) Om $r_1, r_2 \in \mathbb{R}$, $r_1 \neq r_2$

Lösning $y = Ae^{r_1 x} + Be^{r_2 x}$

b) Om $r_1 = r_2 \in \mathbb{R}$, $r_1 = r_2 = r$

$y = Ae^{rx} + Bxe^{rx}$

Om $r = a \pm bi$ · (dvs to komplekse rötter)

$y = Ae^{ax}\cos bx + Be^{ax}\sin bx$

A. 12.6

a) $y'' + 2y' - 5y = 0$, $r^2 + 2r - 5 = 0$

$$r = \frac{-2 \pm \sqrt{4+20}}{2} = \frac{-2 \pm 2\sqrt{6}}{2} = -1 \pm \sqrt{6}$$

$$y(x) = Ae^{(-1+\sqrt{6})x} + Be^{(-1-\sqrt{6})x}$$

b) $y'' - \frac{1}{2}y' - 2y = 0$, $r^2 - \frac{1}{2}r - 2 = 0$

$$r = \frac{\frac{1}{2} \pm \sqrt{\frac{1}{4} + 8}}{2} = \frac{\frac{1}{2} \pm \sqrt{\frac{33}{4}}}{2} = \frac{1 \pm \sqrt{33}}{4}$$

$$y(x) = Ce^{\frac{1-\sqrt{33}}{4} \cdot x} + De^{\frac{1+\sqrt{33}}{4}x}$$

c) $y'' + 4y' + 4y = 0$

$r^2 + 4r + 4 = 0$, $r = \frac{-4 \pm \sqrt{16-16}}{2} = -2$

(én rot) $y(x) = Ce^{-2x} + Dxe^{-2x}$

12.7

a) $2y'' - 7y' + 3y = 0$, $y'' - \frac{7}{2}y' + \frac{3}{2}$

$r^2 - \frac{7}{2}r + \frac{3}{2} = 0$

$r = \frac{\frac{7}{2} \pm \sqrt{\frac{49}{4} - \frac{24}{4}}}{2} = \frac{7 \pm \sqrt{25}}{4} = \begin{cases} 3 \\ \frac{1}{2} \end{cases}$

$\underline{y(x) = Ce^{\frac{1}{2}x} + De^{3x}}$

b) $y'' + 4y' = 0$

$r^2 - 4r = 0$, $r = 0$, $r = 4$

$y(x) = C(e^{0 \cdot x}) + De^{4x} = \underline{\underline{C + De^{4x}}}$

c) $y'' - 10y' + 25y = 0$

$r^2 - 10r + 25 = 0$, $r = \frac{10 \pm \sqrt{100-100}}{2} = 5$

$\underline{y(x) = Ce^{5x} + Dxe^{5x}}$

d) $y'' - 4y' + 5y = 0$

$r^2 - 4r + 5 = 0$, $r = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm \sqrt{-4}}{2}$

$= \frac{4 \pm 2\sqrt{-1}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$ ($a = 2$, $b = 1$)

$y(x) = Ce^{2x}\cos x + De^{2x}\sin x$

12. 9

a) $y' - y = 4$, $y' + f(x)y = g(x)$, $f(x) = -1$, $g(x) = 4$

$y(0) = 2$, $F(x) = \int f(x)dx = \int -dx = -x$

$e^{F(x)} = e^{-x}$, $\int e^{F(x)} g(x)dx =$

$= \int e^{-x} \cdot 4dx = -4e^{-x} + C$

Hadde $y(x) = e^{-F(x)}\left(\int e^{F(x)} g(x)dx + C\right)$

$y(x) = e^{x}(-4e^{-x} + C) = -4 + Ce^{x}$

$y(0) = -4 + C = 2$, $C = 6$

$\underline{\underline{y(x) = -4 + 6e^{x}}}$

b) $y' + xy = x$, $f(x) = x = g(x)$, $\underline{y(0) = 0}$

$F(x) = \int x dx = \frac{1}{2}x^2$, $\int e^{F(x)} g(x)dx =$

$= \int e^{\frac{1}{2}x^2} \cdot x dx = e^{\frac{1}{2}x^2} + C$, $y(x) = e^{-\frac{1}{2}x^2}(e^{\frac{1}{2}x^2} + C)$

$= 1 + Ce^{-\frac{1}{2}x^2}$, $\underline{y(0) = 1 + C = 0, \ C = -1}$

$\underline{\underline{y(x) = 1 - e^{-\frac{1}{2}x^2}}}$

12.11

a) $y'' - 5y' + 4y = 0$, $y(0) = 0$, $y'(0) = 2$

$r^2 - 5r + 4 = 0$, $r = \frac{5 \pm \sqrt{25-16}}{2} = \frac{5 \pm 3}{2} = \begin{cases} 4 \\ 1 \end{cases}$

$y(x) = Ce^{4x} + De^{x}$

$y(0) = 0 = C + D \longrightarrow D = -C$

$y'(x) = 4Ce^{4x} + De^{x}$

$y'(0) = 4C + D = 2$

$4C - C = 2$, $C = \frac{2}{3}$, $D = -\frac{2}{3}$

$y(x) = \frac{2}{3}e^{4x} - \frac{2}{3}e^{x}$

b) $y'' + 3y' + 3y = 0$, $y(0) = 0$, $y'(0) = \sqrt{3}$

$r^2 + 3r + 3 = 0$, $r = \frac{-3 \pm \sqrt{9-12}}{2} = \frac{-3 \pm \sqrt{3}i}{2}$

$y(x) = C e^{-\frac{3}{2}x} \cos(\frac{\sqrt{3}}{2}x) + D e^{-\frac{3}{2}x} \sin(\frac{\sqrt{3}}{2}x)$

$y(0) = C = 0$, $y(x) = D e^{-3/2 x} \sin(\frac{\sqrt{3}}{2}x)$

$y'(x) = -\frac{3}{2} D e^{-3/2 x} \sin(\frac{\sqrt{3}}{2}x) + \frac{\sqrt{3}}{2} D e^{-\frac{3x}{2}} \cos \frac{\sqrt{3}}{2}x$

$y'(0) = \frac{\sqrt{3}}{2} D = \sqrt{3}$, $D = 2$

$y(x) = 2 e^{-3/2 x} \sin(\frac{\sqrt{3}}{2}x)$

c) $y'' - 4y' + 4y = 0$, $y(0) = 0$, $y'(0) = 1$

$r^2 - 4r + 4 = 0$, $r = \frac{4 \pm \sqrt{16-16}}{2} = 2$

$y(x) = C e^{2x} + D x e^{2x}$, $y(0) = C = 0$

$y(x) = D x e^{2x}$, $y'(x) = D e^{2x} + 2 D x e^{2x}$

$y'(0) = D = 1$, $\underline{y(x) = x e^{2x}}$

$\to$ d) $y'' + 2y = 0$, $y(0) = 1$, $y'(0) = 1$

$r^2 + 2 = 0$, $r^2 = -2$, $r = \pm\sqrt{-2} = \pm\sqrt{2}\,i$

$\to y(x) = C\cos\sqrt{2}x + D\sin\sqrt{2}x$

$y(0) = C = 1$, $y(x) = \cos\sqrt{2}x + D\sin\sqrt{2}x$

$y'(x) = -\sqrt{2}\sin\sqrt{2}x + \sqrt{2}D\cos\sqrt{2}x$

$y'(0) = \sqrt{2}D = 1$, $D = \frac{1}{\sqrt{2}}$

$$\underline{\underline{y(x) = \cos\sqrt{2}x + \frac{1}{\sqrt{2}}\sin\sqrt{2}x}}$$