A 11: 3, 4, 5, 6

B 1.10, B 1.12

B 1.12

a)

A skandale avis, B moderat seriøs, C svært seriøs.

x_n, y_n, z_n salgstall for henholdsvis A, B, C

i året $5n$, $n = 0, 1, 2, \dots$

A, B, C får 10% nye kjøpere som ikke hadde avis før.

10% av A slutter går over til B.

5% av B — " — A

$$x_{n+1} = (1{,}1)x_n - 0{,}1x_n + 0{,}05y_n = \underline{x_n + 0{,}05y_n}$$

10% av B slutter og går over til C

5% av C — " — B

5% av B og C slutter og kjøper ikke ny avis.

$$y_{n+1} = 0.1x_n + 1{,}1y_n - 0{,}05y_n - 0{,}1y_n - 0{,}05y_n + 0{,}05z_n$$
$$= \underline{0{,}1x_n + 0{,}9y_n + 0{,}05z_n}$$

$$z_{n+1} = 0{,}1y_n + 1{,}1z_n - 0{,}05z_n - 0{,}05z_n$$
$$= \underline{0{,}1y_n + z_n}, \quad \begin{bmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{bmatrix} = M \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix}, \quad M = ?$$

$$\underline{\underline{M = \begin{bmatrix} 1 & 0{,}05 & 0 \\ 0{,}1 & 0{,}9 & 0{,}05 \\ 0 & 0{,}1 & 1 \end{bmatrix}}}$$

b) Endrer på M og setter

$$M = \begin{bmatrix} 1 & \frac{1}{10} & 0 \\ \frac{1}{10} & 1 & \frac{1}{10} \\ 0 & \frac{1}{10} & 1 \end{bmatrix}$$

. Egenverdier og egenvektoren er ?

Egenverdier

$$\det(M - \lambda I) = \begin{vmatrix} 1-\lambda & \frac{1}{10} & 0 \\ \frac{1}{10} & 1-\lambda & \frac{1}{10} \\ 0 & \frac{1}{10} & 1-\lambda \end{vmatrix} =$$

$$= (1-\lambda)\begin{vmatrix} 1-\lambda & \frac{1}{10} \\ \frac{1}{10} & 1-\lambda \end{vmatrix} - \frac{1}{10}\begin{vmatrix} \frac{1}{10} & \frac{1}{10} \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\left((1-\lambda)^2 - \frac{1}{100}\right)$$

$$-\frac{1}{10}\left(\frac{1}{10}(1-\lambda)\right) = (1-\lambda)\left((1-\lambda)^2 - \frac{2}{100}\right) = 0$$

Egenverdier $\lambda = 1$, og når

$$(1-\lambda)^2 - \frac{2}{100} = 0, \quad (1-\lambda)^2 = \frac{2}{100}, \quad 1-\lambda = \pm\sqrt{\frac{2}{100}} = \pm\frac{\sqrt{2}}{10}$$

$\lambda = 1 \mp \frac{\sqrt{2}}{10}$, Egenverdier: $\lambda = 1$, $\lambda = 1 - \frac{\sqrt{2}}{10}$, $\lambda = 1 + \frac{\sqrt{2}}{10}$

Egenvektorer:

$$\lambda = 1 \ , \quad M \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1 \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\left.\begin{aligned} x + \tfrac{1}{10} y \quad &= x \\ \tfrac{x}{10} + y + \tfrac{1}{10} z &= y \\ \tfrac{1}{10} y + z &= z \end{aligned}\right\}$$

$$\left.\begin{aligned} \frac{y}{10} &= 0 \\ \frac{x}{10} + \frac{z}{10} &= 0 \\ \frac{y}{10} &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} y &= 0 \\ z &= -x \end{aligned}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad t \neq 0$$

$$\lambda = 1 - \frac{\sqrt{2}}{10}, \; M\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \left(1 - \frac{\sqrt{2}}{10}\right)\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$x + \frac{y}{10} = \left(1 - \frac{\sqrt{2}}{10}\right) x$$

$$\frac{x}{10} + y + \frac{z}{10} = \left(1 - \frac{\sqrt{2}}{10}\right) y$$

$$\frac{y}{10} + z = \left(1 - \frac{\sqrt{2}}{10}\right) z$$

$$\left.\begin{aligned} \frac{\sqrt{2}}{\cancel{10}} x + \frac{y}{\cancel{10}} &= 0 \\ \frac{x}{\cancel{10}} + \frac{\sqrt{2}}{\cancel{10}} y + \frac{z}{\cancel{10}} &= 0 \\ \frac{y}{\cancel{10}} + \frac{\sqrt{2}}{\cancel{10}} z &= 0 \end{aligned}\right\}$$

$$x = -\frac{1}{\sqrt{2}} y, \quad z = -\frac{1}{\sqrt{2}} y$$

(passer inn i likning 2)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 1 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = t \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}, \; t \neq 0$$

$s \neq 0$ der $s = -\sqrt{2}\, t$

$\lambda = 1 + \frac{\sqrt{2}}{10}$, Vi får då

$$x + \frac{y}{10} = \left(1 + \frac{\sqrt{2}}{10}\right)x$$

$$\frac{x}{10} + y + \frac{z}{10} = \left(1 + \frac{\sqrt{2}}{10}\right)y$$

$$\frac{y}{10} + z = \left(1 + \frac{\sqrt{2}}{10}\right)z$$

$$\left.\begin{aligned} -\sqrt{2}x + y &= 0 \\ x - \sqrt{2}y + z &= 0 \\ y - \sqrt{2}z &= 0 \end{aligned}\right\}$$

$x = \frac{y}{\sqrt{2}}$, $z = \frac{y}{\sqrt{2}}$ (passer inn i likning 2)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = t \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}, \quad s = \sqrt{2}t, \quad t \neq 0$$

c) Gitt $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$

Finn $\begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix}$. Vil finne a, b, c

slik at $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \underbrace{a \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}}_{\text{(egenvektorene for } M\text{)}}$

$$\left.\begin{aligned} a+b+c &= x_0 \\ -\sqrt{2}b+\sqrt{2}c &= y_0 \\ -a+b+c &= z_0 \end{aligned}\right\} \qquad \begin{bmatrix} 1 & 1 & 1 & x_0 \\ 0 & -\sqrt{2} & \sqrt{2} & y_0 \\ -1 & 1 & 1 & z_0 \end{bmatrix} R_1 \text{ til } R_3 \sim$$

$$\begin{bmatrix} 1 & 1 & 1 & x_0 \\ 0 & -\sqrt{2} & \sqrt{2} & y_0 \\ 0 & 2 & 2 & x_0+z_0 \end{bmatrix} -\frac{1}{\sqrt{2}}R_2 \sim \begin{bmatrix} 1 & 1 & 1 & x_0 \\ 0 & 1 & -1 & -\frac{y_0}{\sqrt{2}} \\ 0 & 2 & 2 & x_0+z_0 \end{bmatrix} -2R_2 \text{ til } R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & x_0 \\ 0 & 1 & -1 & \frac{-y_0}{\sqrt{2}} \\ 0 & 0 & 4 & x_0+\sqrt{2}y_0+z_0 \end{bmatrix} \frac{1}{4}R_3 \sim \begin{bmatrix} 1 & 1 & 1 & x_0 \\ 0 & 1 & -1 & \frac{-y_0}{\sqrt{2}} \\ 0 & 0 & 1 & \frac{1}{4}(x_0+\sqrt{2}y_0+z_0) \end{bmatrix}$$

$c = \frac{1}{4}(x_0+\sqrt{2}y_0+z_0), \quad b-c = \frac{-y_0}{\sqrt{2}}$

$b = c - \frac{y_0}{\sqrt{2}} = \frac{1}{4}x_0 - \frac{1}{4}y_0\sqrt{2} + \frac{z_0}{4}, \quad a+b+c = x_0$

$a = -b-c+x_0 = \frac{x_0-z_0}{2}$

$$\begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} = M^n \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = M^n \left(a \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} \right)$$

$$= a M^n \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b M^n \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} + c M^n \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$$

$$= a \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b\left(1-\frac{\sqrt{2}}{10}\right)^n \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} + c\left(1+\frac{\sqrt{2}}{10}\right)^n \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$$ der a, b, c er gitt over.

Anta $(x_0, y_0, z_0) \neq (0,0,0)$

Skal vise at

$$\lim_{n\to\infty} \frac{x_n}{y_n} = k_1, \quad \lim_{n\to\infty} \frac{y_n}{z_n} = k_2$$

k_1, k_2 er uavhengig av (x_0, y_0, z_0)

$$\frac{x_n}{y_n} = \frac{a + b\left(1-\frac{\sqrt{2}}{10}\right)^n + c\left(1+\frac{\sqrt{2}}{10}\right)^n}{-b\left(1-\frac{\sqrt{2}}{10}\right)^n\sqrt{2} + c\left(1+\frac{\sqrt{2}}{10}\right)^n\sqrt{2}} =$$

$$= \frac{\overbrace{\left(a/\left(1+\frac{\sqrt{2}}{10}\right)^n\right)}^{\to 0} + b\overbrace{\left(\left(1-\frac{\sqrt{2}}{10}\right)/\left(1+\frac{\sqrt{2}}{10}\right)\right)^n}^{\to 0} + c}{-\sqrt{2}\underbrace{\left(\left(1-\frac{\sqrt{2}}{10}\right)/\left(1+\frac{\sqrt{2}}{10}\right)\right)^n}_{0} b + \sqrt{2}c} \xrightarrow[n\to\infty]{} \frac{c}{\sqrt{2}c} = \frac{1}{\sqrt{2}} = \underline{\frac{1}{2}\sqrt{2}}$$

$$\frac{y_n}{z_n} = \frac{-b\left(1-\frac{\sqrt{2}}{10}\right)^n\sqrt{2} + c\left(1+\frac{\sqrt{2}}{10}\right)^n\sqrt{2}}{-a + b\left(1-\frac{\sqrt{2}}{10}\right)^n + c\left(1+\frac{\sqrt{2}}{10}\right)^n} =$$

$$= \frac{-b\sqrt{2}\overbrace{\left(\left(1-\frac{\sqrt{2}}{10}\right)/\left(1+\frac{\sqrt{2}}{10}\right)\right)^n}^{0} + \sqrt{2}c}{\underbrace{\frac{-a}{\left(1+\frac{\sqrt{2}}{10}\right)^n}}_{\to 0} + \underbrace{b\left(\frac{1-\frac{\sqrt{2}}{10}}{1+\frac{\sqrt{2}}{10}}\right)^n}_{\to 0,\ n\to\infty} + c} \xrightarrow[n\to\infty]{} \frac{\sqrt{2}c}{c} = \underline{\sqrt{2}}$$

N.B $c \neq 0$ siden

$c = \frac{1}{4}(x_0 + \sqrt{2}y_0 + z_0)$

og $(x_0, y_0, z_0) \neq 0$

Grensen avhenger ikke av a, b, c og derfor ikke av x_0, y_0, z_0

A 11

3) a) $f(x) = \ln|\cos x| + \sin\left(\frac{x}{2}\right)$

$\cos x \neq 0$, $\left(\ln|x|\right)' = \frac{1}{x}$)

$$f'(x) = \frac{1}{\cos x}(\cos x)' + \cos\left(\frac{x}{2}\right)\left(\frac{x}{2}\right)'$$

$$= \frac{-\sin x}{\cos x} \cancel{+} \frac{1}{2}\cos\left(\frac{x}{2}\right) = -\tan x + \frac{1}{2}\cos\frac{x}{2}$$

b) $f(x) = \dfrac{4x^3+5x}{\sin(x+2)}$ $= \dfrac{(12x^2+5)\sin(x+2) - (4x^3+5x)\cos(x+2)}{\sin^2(x+2)}$

$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

c) $f(x) = \dfrac{x}{|x|}$, $x \neq 0$, $f(x) = \begin{cases} \frac{x}{x} = 1 & \text{når } x > 0 \\ \frac{-x}{x} = -1 & \text{når } x < 0 \end{cases}$

siden $(k)' = 0$ der k er konstant må $f'(x) = 0$

Alternativ $(|x|)' = \dfrac{|x|}{x}$

$$f'(x) = \frac{(x)'|x| - x(|x|)'}{|x|^2} = \frac{|x| - x\frac{|x|}{x}}{x^2} = \frac{|x| - |x|}{x^2} = 0$$

11.4

a) $f_1(x) = \sin x \cos x + x^2$ $\qquad (uv)' = u'v + uv'$

$f_1'(x) = \cos x \cos x + \sin x(-\sin x) + 2x$

$= \cos^2 x - \sin^2 x + 2x \left(= \cos 2x + 2x\right)$

b) $f_2(x) = \cos(\sqrt{x}) - \sqrt{x}$ $\quad , \quad (x^a)' = a x^{a-1}$

$\sqrt{x} = x^{\frac{1}{2}}$

$f_2'(x) = -\sin(\sqrt{x})\frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{-1}{2}\frac{1}{\sqrt{x}}(\sin\sqrt{x}+1)$

c) $f_3'(x) = \left(\frac{\sin(x^2)}{3x}\right)' = \frac{\cos(x^2)2x \cdot 3x - \sin(x^2)3}{9x^2} = \frac{6x^2\cos x^2 - 3\sin x^2}{9x^2}$