Eksamen høst. 2014
oppgave 5
Midtveiseksamen 2015. oppgave 4 $\longrightarrow$
Eksamen høst 2010.

<u>2014, 5</u>

$\longrightarrow v(t)$ $\longleftarrow$ motstandskraft

fly $\longrightarrow F$ skyvekraft   proporsjonal med $v(t)^2$

$-kv^2$

$t=0$ flyet masse m_0, $m = m_0 - t = m(t)$

Newtons 2. lov $\longrightarrow$ <u>m·akselrasjon = kraft</u>

akselrasjon $= v'(t)$ gir oss

$$\underline{mv' = F - kv^2}$$

5a) Anta $F=4$, $k=1$, $m_0=1$, $v(0)=0$

$v(t) = ?$

$$(m_0 - t)v' = F - kv^2 = 4 - v^2$$

$$(1-t)v' = 4 - v^2$$

$$\frac{v'}{4-v^2} = \frac{1}{1-t}, \quad \int \frac{dv}{4-v^2} = \int \frac{1}{(1-t)}dt$$

$$\frac{1}{4-v^2} = \frac{A}{2-v} + \frac{B}{2+v} = \frac{(A-B)v + 2(A+B)}{4-v^2}$$

må ha $A-B=0$, $2(A+B)=1$

$B=A$, $4A=1$, $A=\frac{1}{4}$

$$\int \frac{dv}{4-v^2} = \frac{1}{4}\left(\int\left(\frac{1}{2-v} + \frac{1}{2+v}\right)dv\right) =$$

$$= \frac{1}{4}\left(-\ln|2-v| + \ln|2+v|\right)$$

$$= \frac{1}{4}\ln\left|\frac{2+v}{2-v}\right| = \int \frac{dt}{1-t} = -\ln|1-t| + C$$

$$\ln\left|\frac{2+v}{2-v}\right| = -4\ln|1-t| + C$$

$$\left|\frac{2+v}{2-v}\right| = e^C e^{-4\ln|1-t|} = \frac{e^C}{|1-t|^4}$$

$$\frac{2+v}{2-v} = \pm\frac{e^C}{(1-t)^4} = K\frac{1}{(1-t)^4}, \quad (K = \pm e^C)$$

$v(0)=0$, $\frac{2}{2} = 1 = K$, $\frac{2+v}{2-v} = \frac{1}{(1-t)^4}$

$$(2+v) = \frac{2-v}{(1-t)^4} \Rightarrow v + v\frac{1}{(1-t)^4} = \frac{2}{(1-t)^4} - 2$$

$$v\left(1 + \frac{1}{(1-t)^4}\right) = 2\left(\frac{1}{(1-t)^4} - 1\right)$$

$$v = 2\left(\frac{\left(\frac{1}{(1-t)^4} - 1\right)}{\left(1 + \frac{1}{(1-t)^4}\right)}\right) = \underline{\underline{2\,\frac{1-(1-t)^4}{1+(1-t)^4}}}$$

5b) Antar at brennstoffet $= \frac{m_0}{2} = \frac{1}{2}$ ved $t = 0$

$m(t) = m_0 - t = 1 - t$, ser at når $t = \frac{1}{2}$

$m = \frac{1}{2}$, dvs. flyet har kvittet seg med all brennstoff. Da blir $F = 0$, $m \equiv 1/2$

(dvs. masse av fly en konstant lik $\frac{1}{2}$)

Newton's 2. lov gir nå

⊛ $\frac{1}{2} v' = -v^2$ (siden $F=0$) gjelder for $t > \frac{1}{2}$

$v(t) = ?$ for $t > \frac{1}{2}$. Finner generell løsning av ⊛

$\frac{1}{2} \frac{v'}{v^2} = -1$, $\int \frac{1}{2} \frac{dv}{v^2} = \int -dt =$

$-\frac{1}{2} \frac{1}{v} = -t + C$, $\frac{1}{v} = 2t + C$

$v = \frac{1}{2t + C}$, Hadde at når $t \leq \frac{1}{2}$

$v(t) = 2 \frac{1-(1-t)^4}{1+(1-t)^4}$, $v(\frac{1}{2}) = 2 \frac{1 - \frac{1}{16}}{1 + \frac{1}{16}} = \frac{30}{17}$

må også ha $v(\frac{1}{2}) = \frac{1}{1+C} = \frac{30}{17}$

$1 + C = \frac{17}{30}$, $C = \frac{17}{30} - 1 = -\frac{13}{30}$

$$\underline{\underline{v(t) = \frac{1}{2t - \frac{13}{30}} = \frac{30}{60t - 13}}} \quad \text{for } t \geq \frac{1}{2}.$$

Midtveis eksamen 2015

9) Realdel av $(2-i)(1+i)$ er ?

$(2-i)(1+i) = 2 - i\cdot i - i + 2i = 2+1+i = 3+i$

Realdel $= 3$ ((a))

10) $\frac{1}{3} e^{i\frac{\pi}{4}} \cdot 3e^{-i\frac{\pi}{4}} = 1\, e^{i(\frac{\pi}{4}-\frac{\pi}{4})} = 1e^{0\cdot i} = 1$

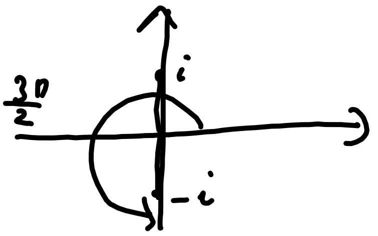
(e)

11) Normalform $z = \sqrt{2}\, e^{i\frac{\pi}{4}} - 1 = ?$

$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) - 1 =$

$= \sqrt{2}\left(\frac{1}{2}\sqrt{2} + i\frac{1}{2}\sqrt{2}\right) - 1 = 1+i-1 = i$ ((c))

12) Polarform $z = -2i$ er



$z = 2(-i) = 2e^{\frac{3\pi}{2}i}$ (a))

13) $x_{n+1} - \frac{1}{3}x_n = 0$ $\quad x_3 = ?$

Vet $x_n = C\left(\frac{1}{3}\right)^n$, $C = ?$ må vite x_0 for å finne C (e))

14) $x_{n+1} - 3x_n = -2$, $x_0 = 1$

$x_{20} = ?$ $\quad x_n^h = C \cdot 3^n$, $x_n^s = A$, $A - 3A = -2$

$-2A = -2, A = 1$

$x_n = C \cdot 3^n + 1$, generell lösning

$x_0 = C + 1 = 1$, $C = 0$, $x_n = 1$, $x_{20} = 1$

(c))

15) $x_{n+1} - x_n = 2n+1$, $x_0 = 0$, $x_{10} = ?$

$x_n = C \cdot 1^n = C$ lösning av homogene.

Speciell lösning, $An^2 + Bn$. (sedan x_n = konstant är lösning av den homogene)

$\left(A(n+1)^2 + B(n+1)\right) - (An^2 + Bn)$

$= (\cancel{An^2} + 2An + A + \cancel{Bn} + B) - (\cancel{An^2} + \cancel{Bn})$

$= 2An + A + B = 2n + 1$, $\quad 2A = 2$, $A = 1$, $A + B = 1 + B = 1$

$B = 0$, $\quad x_n = C + n^2$, $\quad x_0 = C = 0$, $\quad x_n = n^2$

$x_{10} = (10)^2 = 100$ (e))

(16)

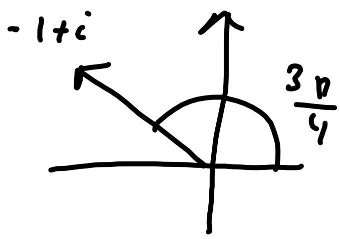
$$x_{n+2} + 2x_{n+1} + 2x_n = 0 \quad , \quad x_0 = 0, \; x_1 = 1$$

$$\lambda^2 + 2\lambda + 2 = 0 \; , \quad \lambda = \frac{-2 \pm \sqrt{4-8}}{2} =$$

$$= \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = \underline{-1 \pm i} = r e^{i\theta}$$

om $\lambda = a \pm bi$,

$$x_n = r^n (A\cos n\theta + B\sin n\theta)$$



$$|1+i| = \sqrt{1+1} = \sqrt{2}$$

$$\lambda = \sqrt{2}\, e^{i\frac{3\pi}{4}},$$

$$x_n = (\sqrt{2})^n \left((\cos n\tfrac{3\pi}{4}) A + B \sin (n\tfrac{3\pi}{4}) \right)$$

$$x_0 = (\sqrt{2})^0 A = A = 0$$

$$x_1 = \sqrt{2} \left(B \sin \frac{3\pi}{4} \right) = \sqrt{2}\, \frac{1}{2}\sqrt{2} = 1$$

$$B = 1, \quad x_n = (\sqrt{2})^n \sin \frac{n3\pi}{4} \; , \quad x_{20} = ?$$

$$x_{20} = (\sqrt{2})^{20} \sin \frac{60\pi}{4} = (\sqrt{2})^{20} \sin (15\pi) = 0$$

(b)

17) $X_{n+2} - 3X_{n+1} + 2X_n = -2n+1$

$X_0 = 0,\ X_1 = 1,\ X_{20} = ?$

$\lambda^2 - 3\lambda + 2 = 0,\ \lambda = \frac{3 \pm \sqrt{9-8}}{2} = \begin{cases} 2 \\ 1 \end{cases}$

$X_n^h = C \cdot 2^n + D,\quad X_n^s = An^2 + Bn$

$\left(A(n+2)^2 + B(n+2)\right) - 3\left(A(n+1)^2 + B(n+1)\right) + 2(An^2 + Bn) =$

$= (\cancel{An^2} + 4An + 4A + Bn + 2B) - 3(\cancel{An^2} + 2An + A + Bn + B)$

$+ (\cancel{2An^2} + 2Bn) = -2An + A - B = -2n + 1$

$A = 1,\ B = 0,\ X_n = C \cdot 2^n + D + n^2$

$X_0 = 0,\ X_1 = 1,\quad \begin{cases} X_0 = C + D = 0 \\ X_1 = 2C + D + 1 = 1 \end{cases} \quad \begin{cases} C + D = 0 \\ 2C + D = 0 \end{cases} \Rightarrow C = 0,\ D = 0$

$X_n = n^2,\ X_{20} = (20)^2 = 400$ (d)).

Eksamen høst 2010

Oppgave 1

$$y'' - 4y' + 13y = 0$$

a) Finn generell løsning

$$r^2 - 4r + 13 = 0, \quad r = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm \sqrt{-36}}{2}$$

$$= \frac{4 \pm 6\sqrt{-1}}{2} = \underline{2 \pm 3i}$$

$$y(x) = Ce^{2x}\cos 3x + De^{2x}\sin 3x$$

b) Har nå $y(0) = 1$, $y'(0) = -1$

$$y(0) = C = 1, \quad y(x) = e^{2x}\cos 3x + De^{2x}\sin 3x$$

$$y'(x) = 2e^{2x}\cos 3x - 3e^{2x}\sin 3x + 2De^{2x}\sin 3x + 3De^{2x}\cos 3x$$

$$y'(0) = 2 + 3D = -1, \quad 3D = -3, \quad D = -1$$

$$y(x) = \underline{\underline{e^{2x}\cos 3x - e^{2x}\sin 3x}}$$

① Oppgave 2

a) Finn alle antideriverte til
$f(x) = x\cos x$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx =$$

$v' = \cos x \quad u = x$

$v = \sin x \quad u' = 1$

$$= x\sin x - (-\cos x) + C$$

$$= \underline{\underline{x\sin x + \cos x + C}}$$

b)

$xy' + y = x\cos x, \quad y\left(\frac{\pi}{2}\right) = 3$

$y' + \frac{y}{x} = \cos x$, anta at $x > 0$

$f(x) = \frac{1}{x}, \quad \int f(x)\,dx = \ln x = F(x)$

$e^{F(x)} = x, \quad \int e^{F(x)} \underbrace{\cos x}_{g}\,dx = \int x\cos x\,dx \overset{(a)}{=} x\sin x + \cos x + C$

$y = e^{-F(x)}\left(\text{—"—} \right) =$

$= \frac{1}{x}(x\sin x + \cos x + C) = \sin x + \frac{\cos x}{x} + \frac{C}{x}$

$y\left(\frac{\pi}{2}\right) = 1 + \frac{C}{\frac{\pi}{2}} = 1 + \frac{2C}{\pi} = 3, \quad \frac{2C}{\pi} = 2, \quad 2C = 2\pi, \quad C = \pi$

$$\underline{\underline{y(x) = \sin x + \frac{\cos x + \pi}{x}}}$$