A11: 1, 3, 5, 7, 8, 9, 10
A12: 2

11.1: deriver funksjonene

a) $f(x) = x^2(x^3+2)$.

Skriv ut: $f(x) = x^2 \cdot x^3 + x^2 \cdot 2$
$= x^5 + 2x^2$

$f'(x) = 5x^4 + 2 \cdot 2x$
$= 5x^4 + 4x$

n vilkårlig tall i $\mathbb{Q}$.

[Husk: $(x^n)' = nx^{n-1}$]

c) $h(x) = 3\ln\left(\frac{1}{x}\right) - \pi$
$= -3\ln(x) - \pi$

$h'(x) = -3 \cdot \frac{1}{x} - 0$ konstant

$h'(x) = \frac{-3}{x}$

$\ln(a^n) = n\ln(a)$.
$n = -1$: $\ln(a^{-1}) = -\ln(a)$
$\ln\left(\frac{1}{a}\right) = -\ln(a)$.

$\ln(x)' = \frac{1}{x}$.

d) $k(x) = \dfrac{\overbrace{x^2 - 7x + 2}^{u}}{\underbrace{x^3 - 1}_{v}}$

Brøkregel: $\left(\frac{u(x)}{v(x)}\right)' = \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

$u = x^2 - 7x + 2$ $\quad v = x^3 - 1$
$u' = 2x - 7$ $\quad v' = 3x^2$

$k'(x) = \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} = \frac{(2x-7)(x^3-1) - (x^2-7x+2)(3x^2)}{(x^3-1)^2}$

$= \frac{2x^4 - 2x - 7x^3 + 7 - 3x^4 + 21x^3 - 6x^2}{(x^3-1)^2}$

$= \frac{-x^4 + 14x^3 - 6x^2 - 2x + 7}{(x^3-1)^2}$

e) $m(x) = \overset{u}{x}\,\overset{v}{e^{x^2}}$

Produktregelen: $(uv)' = u'v + uv'$

$u = x$
$u' = 1$

$v = e^{x^2}$
$v(x) = e^{x^2}$
$\boxed{v(t) = e^t.}$

Kjerneregel:
Kjerne: $t = x^2$

Kjerneregel:
$f(u(x))' = f'(u(x)) \cdot u'(x)$

$v'(t) = e^t$
$t(x) = x^2$
$t'(x) = 2x$.

$(v(x))' = v'(t(x)) \cdot t'(x) = e^{t(x)} \cdot 2x = e^{x^2} \cdot 2x = 2xe^{x^2}$

$m(x) = xe^{x^2}$.
$m'(x) = u'v + uv' = 1 \cdot e^{x^2} + x \cdot (2xe^{x^2})$
$= e^{x^2} + 2x^2e^{x^2}$
$= e^{x^2}(1 + 2x^2)$

11.3 a) $f(x) = \underbrace{\ln|\cos(x)|} + \sin\left(\frac{x}{2}\right)$, $[\cos(x) \neq 0]$

kjerne: $u(x) = |\overbrace{\cos(x)}^{v(x)}|$.

Ny kjerne: $v(x) = \cos(x) \Rightarrow v'(x) = -\sin(x)$

$u(x) = |v|$

$\ln|\cos(x)|$

$= \ln(u)$

Kjerneregel: $u'(x) = \underbrace{|v|'}_{\text{mhp } v} \cdot v'(x)$

$$\boxed{\begin{aligned} u'(x) &= \frac{|v|}{v} \cdot v'(x) \\ &= \frac{|\cos(x)|}{\cos(x)}(-\sin(x)) \\ &= \frac{-\sin(x)|\cos(x)|}{\cos(x)} \end{aligned}}$$

$$\boxed{\text{Husk: } |x|' = \frac{|x|}{x}}$$

$\ln|\cos(x)| = \ln(u(x))$

$$\begin{aligned} (\ln|\cos(x)|)' &= \frac{1}{u(x)} \cdot u'(x) \\ &= \frac{1}{|\cos(x)|} \cdot \left(\frac{-\sin(x)|\cos(x)|}{\cos(x)}\right) \\ &= \frac{-\sin(x)\cancel{|\cos(x)|}}{\cancel{|\cos(x)|}\cos(x)} \\ &= \frac{-\sin(x)}{\cos(x)} = \boxed{-\tan(x)}. \end{aligned}$$

$[\ln(x)' = \frac{1}{x}]$

$f(x) = \underbrace{\ln|\cos(x)|} + \underbrace{\sin\left(\frac{x}{2}\right)}$, $[\cos(x) \neq 0]$

lineært uttrykk

$\left(\sin\left(\frac{x}{2}\right)\right)' = \frac{1}{2}\cos\left(\frac{x}{2}\right)$

Litt mer generelt: $(f(\underbrace{ax+b}_{\text{lin. uttrykk i } x}))' = af'(ax+b)$ (kjerneregel)

Setter sammen: $\underline{\underline{f'(x) = -\tan(x) + \frac{1}{2}\cos\left(\frac{x}{2}\right)}}$

1.5 a) $f(x) = \cos(x^2)$: kjerne $u = x^2$

$u' = 2x$

$f(x) = \cos(u)$

$f'(x) = -\sin(u) \cdot u'$

$= -\sin(x^2) \cdot 2x$

$= -2x \sin(x^2)$

b) $f(x) = \underbrace{4x}_{u}\underbrace{\ln(x)}_{v}$

$u = 4x \quad v = \ln(x)$

$u' = 4 \quad v' = \frac{1}{x}$

$f'(x) = u'v + uv'$

$f'(x) = 4\ln(x) + 4\cancel{x} \cdot \frac{1}{\cancel{x}}$

$f'(x) = 4\ln(x) + 4$

11.7: Avgjør om funksjonane passer:

a) $\underbrace{y''}_{\text{orden 2}} + y = 0$, $y(x) = \cos(x)$.

$y'(x) = -\sin(x)$

$y''(x) = -\cos(x)$

Setter inn: $y''(x) + y(x) = -\cos(x) + \cos(x) = 0$. ✓

b) $\underbrace{y' - y}_{\text{orden 1}} = 20$: $y(x) = 20e^x$

$y'(x) = 20e^x$

$y'(x) - y(x) = 20e^x - 20e^x = 0 \neq 20$ ✗

$\boxed{(e^x)' = e^x}$

11.8 Integrer:

b) $I = \underbrace{\int \left(x^{\frac{5}{6}} - 1\right) dx}_{\text{ubestemt integral}}$

$\boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1} + C}$

$\boxed{\int a\,dx = ax + C}$ a konstant

$I = \int x^{\frac{5}{6}}\,dx + \int (-1)\,dx$

$= \frac{x^{\frac{5}{6}+1}}{\frac{5}{6}+1} + (-1)x + C$

$\frac{11}{6} = \frac{5}{6}+1$

$= \frac{x^{\frac{11}{6}} \cdot 6}{\frac{11}{6} \cdot 6} - x + C = \frac{6x^{\frac{11}{6}}}{11} - x + C$

---

d) $I = \int x^3 \sin(x^4)\,dx$, $u = x^4$

substitusjon:

$u = x^4$

$\frac{du}{dx} = 4x^3$

$du = 4x^3\,dx$

$\frac{du}{4x^3} = dx$

$I = \int x^3 \sin(u) \frac{du}{4x^3}$

$= \int \frac{\cancel{x^3} \sin(u)}{4\cancel{x^3}}\,du$

$= \int \frac{\sin(u)}{4}\,du$

$\boxed{\int \sin(x)\,dx = -\cos(x) + C}$

$= \frac{1}{4}(-\cos(u)) + C$

$= -\frac{1}{4}\cos(x^4) + C$

---

f) $I = \int \frac{1}{(x-1)(x+2)}\,dx$

Delbrøksoppspalting:

$\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2} \quad \Big|\cdot (x-1)(x+2)$

$1 = A(x+2) + B(x-1)$

$1 = Ax + 2A + Bx - B$

$1 = \underbrace{(A+B)}_{=0}x + \underbrace{2A - B}_{=1}$

$L_1: A + B = 0$

$L_2: 2A - B = 1$

$L_1: B = -A$

$L_2: 2A - (-A) = 1$

$2A + A = 1$

$3A = 1$

$A = \frac{1}{3}$

$L_1: B = -\frac{1}{3}.$

$\frac{1}{(x-1)(x+2)} = \frac{\frac{1}{3}}{x-1} - \frac{\frac{1}{3}}{x+2}.$

$I = \int \frac{\frac{1}{3}}{x-1} - \frac{\frac{1}{3}}{x+2}\,dx$

$= \frac{1}{3}\int \frac{1}{x-1}\,dx - \frac{1}{3}\int \frac{1}{x+2}\,dx.$

lin. uttrykk i x.

$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C$

F(x) er en antiderivert til f(x)

$\boxed{\int \frac{1}{x}\,dx = \ln(x) + C}$, $x > 0$

antiderivert til $\frac{1}{x}$

$\int \frac{1}{x}\,dx = \ln|x| + C$

$I = \frac{1}{3}\ln(x-1) - \frac{1}{3}\ln(x+2) + C$

$\left(I = \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + C\right)$

11.9: $I = \int \underbrace{e^x}_{f(x)} \underbrace{\sin(x)}_{g(x)}\,dx$

$f(x) = e^x \quad F(x) = e^x$

$g(x) = \sin(x) \quad g'(x) = \cos(x)$

Delvis integrasjon: $\int f(x)g(x)\,dx = F(x)g(x) - \int F(x)g'(x)\,dx$

F(x) er en anti-derivert til f(x).

$I = e^x \sin(x) - \underbrace{\int e^x \cos(x)\,dx}_{I_1}.$

$I_1 = \int \underbrace{e^x}_{f(x)} \underbrace{\cos(x)}_{g(x)}\,dx \overset{\text{delvis integrasjon}}{=} e^x\cos(x) - \int e^x(-\sin(x))\,dx$

$= e^x\cos(x) + \underbrace{\int e^x \sin(x)\,dx}_{I}$

$I = e^x\sin(x) - I_1$

$I_1 = e^x\cos(x) + I$

$I = e^x\sin(x) - (e^x\cos(x) + I)$

$I = e^x\sin(x) - e^x\cos(x) - I$

$2I = e^x\sin(x) - e^x\cos(x)$

$I = \frac{1}{2}\left(e^x\sin(x) - e^x\cos(x)\right) + C$

$I = \frac{e^x}{2}(\sin(x) - \cos(x)) + C$

b) $I = \int \frac{\ln(x)}{x}\,dx = \int \underbrace{\frac{1}{x}}_{f(x)} \cdot \underbrace{\ln(x)}_{g(x)}\,dx$

$\overset{\text{delvis}}{\underset{\text{integrasjon}}{=}} \ln(x)\ln(x) - \underbrace{\int \ln(x)\cdot\frac{1}{x}\,dx}_{I}$ (anta $x>0$)

$I = \ln(x)^2 - I$

$2I = \ln(x)^2$

$\underline{\underline{I = \frac{1}{2}\ln(x)^2 + C}}$

c) $I = \int \frac{1}{x^2-4}\,dx$

$\frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)}$

Bruk delbrøksoppspalting og se forrige oppgave.

$\underline{11.10}$ a) $I = \int 3\ln(x)\,dx$

$= 3\underbrace{\int \ln(x)\,dx}_{I_1}$

$I_1 = \int \ln(x)\,dx = \int \underset{f(x)}{1} \cdot \underset{g(x)}{\ln(x)}\,dx$ (Delvis integrasjon)

$= x\ln(x) - \int \cancel{x} \cdot \frac{1}{\cancel{x}}\,dx$

$= x\ln(x) - \int 1\,dx$

$I_1 = x\ln(x) - x + C$

$I = 3I_1 = 3x\ln(x) - 3x + 3C$

$= 3x\ln(x) - 3x + C_1$ $\quad | C_1 = 3C |$ ny konstant.

b) $I = \int \cos^2(x)\,dx = \int \underbrace{\cos(x)}_{f(x)} \cdot \underbrace{\cos(x)}_{g(x)}\,dx$

(delvis)

$= \sin(x)\cos(x) - \int \sin(x)(-\sin(x))\,dx$

$= \sin(x)\cos(x) + \underbrace{\int \sin^2(x)\,dx}_{I_1}$

$I_1 = \int \sin^2(x)\,dx$.

$\sin^2(x) + \cos^2(x) = 1$
$\sin^2(x) = 1 - \cos^2(x)$

$I_1 = \int 1 - \cos^2(x)\,dx$

$= x - \underbrace{\int \cos^2(x)\,dx}_{I}$

$I = \sin(x)\cos(x) + I_1$

$I_1 = x - I$

$I = \sin(x)\cos(x) + x - I$

$2I = \sin(x)\cos(x) + x$

$I = \frac{1}{2}(\sin(x)\cos(x) + x) + C$

c) $I = \int x\ln(x^2)\,dx$.

$u = x^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{du}{2x} = dx$

$\int \ln(x)\,dx = x\ln(x) - x + C$ av a)

$I = \int x\ln(u)\frac{du}{2x}$

$= \int \frac{\cancel{x}\ln(u)}{2\cancel{x}}\,du$

$= \int \frac{\ln(u)}{2}\,du$

$= \frac{1}{2}(u\ln(u) - u) + C$

$= \frac{1}{2}(x^2\ln(x^2) - x^2) + C$

$= \frac{x^2}{2}(2\ln(x) - 1) + C$

d) $I = \int x \sin(\sqrt{x})\,dx$

$u = \sqrt{x} = x^{\frac{1}{2}}$

$\frac{du}{dx} = \frac{1}{2}x^{(\frac{1}{2}-1)} = \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{1}{2}\frac{1}{x^{\frac{1}{2}}} = \frac{1}{2}\cdot\frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$du = \frac{dx}{2\sqrt{x}} \quad \Big| \cdot 2\sqrt{x}$

$2\sqrt{x}\,du = dx$

$u = \sqrt{x} \Rightarrow u^2 = x$

$I = \int x \sin(u)\cdot 2\sqrt{x}\,du$

$I = \int u^2 \sin(u)\cdot 2u\,du$

$I = 2\underbrace{\int u^3 \sin(u)\,du}_{I_1}$

$I = 2I_1$

$I_1 = \int u^3 \sin(u)\,du$ — Delvis integrasjon

deriver, integrerer

$= \int \sin(u)\cdot u^3\,du$

$= -\cos(u)\cdot u^3 - \int(-\cos(u))\cdot 3u^2\,du$

$I_1 = -\cos(u)\cdot u^3 + \underbrace{\int \cos(u)\cdot 3u^2\,du}_{I_2}$

$I_1 = -\cos(u)\cdot u^3 + I_2$

$I_2 = \int \cos(u)\cdot 3u^2\,du$ — Delvis integrasjon

$= \sin(u)\cdot 3u^2 - \underbrace{\int \sin(u)\cdot 6u\,du}_{I_3}$

$\boxed{I_2 = \sin(u)\cdot 3u^2 - I_3}$

$I_3 = \int \sin(u)\cdot 6u\,du$

$= -\cos(u)\cdot 6u - \int(-\cos(u))\cdot 6\,du$

$= -\cos(u)\cdot 6u + \int \cos(u)\cdot 6\,du$

$\boxed{I_3 = -\cos(u)6u + \sin(u)\cdot 6}$

(Setter inn konstanten til slutt)

$I = 2I_1, \quad I_1 = -\cos(u)\cdot u^3 + I_2$

$\boxed{I_2 = \sin(u)\cdot 3u^2 - I_3}$

$I = 2\left(-\cos(u)\cdot u^3 + \left(\sin(u)\cdot 3u^2 - \left(-\cos(u)\cdot 6u + \sin(u)\cdot 6\right)\right)\right)$

$= 2\left(-\cos(u)\cdot u^3 + \sin(u)\cdot 3u^2 + \cos(u)\cdot 6u - \sin(u)\cdot 6\right)$

$= 2\left(\cos(u)\left(-u^3 + 6u\right) + \sin(u)\left(3u^2 - 6\right)\right)$

$= 2(-u^3 + 6u)\cos(u) + 2(3u^2 - 6)\sin(u) + C$

$=$ fyll ut selv ... $\quad [u = \sqrt{x}$