12.4 Løs difflikningen:

a) $y' - 2x\sqrt{y} = 0 \quad , \quad y > 0.$

$$y' = 2x\sqrt{y} \quad \Big| \cdot \frac{1}{\sqrt{y}}$$

$$(*) \quad \underbrace{\frac{y'}{\sqrt{y}}}_{\text{bare } y'\text{er}} = \underbrace{2x}_{\text{bare } x'\text{er}} \qquad (\text{Separabel})$$

$$\boxed{p(y)\,y' = g(x)}$$

$$\boxed{y'dx = \frac{dy}{dx}dx = dy.}$$

$$(*) \quad \underbrace{\int \frac{y'}{\sqrt{y}}dx} = \int 2x\,dx$$

$$\underbrace{\int \frac{1}{\sqrt{y}}dy}_{I_1} = \underbrace{\int 2x\,dx}_{I_2}$$

$$I_1 = \int y^{-\frac{1}{2}}dy = \frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = \frac{y^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{y}$$

$$\left[\, y^{-\frac{1}{2}} = \frac{1}{y^{\frac{1}{2}}} = \frac{1}{\sqrt{y}} \,\right]$$

$$I_2 = 2\left(\frac{1}{2}x^2\right) + C = x^2 + C$$

$$\boxed{\int x\,dx = \frac{1}{2}x^2 + C}$$

$$2\sqrt{y} = x^2 + C.$$

$$\sqrt{y} = \frac{x^2 + C}{2}$$

$$y = \left(\frac{x^2+C}{2}\right)^2 = \underline{\underline{\frac{1}{4}(x^2+C)^2}}$$

12.5 a) $y^2 y' = 5x$ (Separabel)

$$\int y^2 \underbrace{y' dx}_{dy} = \int 5x\,dx \qquad (p(y)\,y' = g(x))$$

$$\int y^2 dy = \int 5x\,dx.$$

$$\frac{y^{2+1}}{2+1} = 5 \cdot \frac{1}{2} x^2 + C$$

$$\frac{y^3}{3} = \frac{5}{2}x^2 + C$$

$$y^3 = 3\left(\frac{5}{2}x^2 + C\right)$$

$$y^3 = \frac{15}{2}x^2 + 3C$$

$$\underline{\underline{y = \sqrt[3]{\frac{15}{2}x^2 + 3C}}} \qquad \left(\sqrt[3]{\frac{15}{2}x^2 + D}\right) \text{ også gyldig løsning.}$$

$\boxed{\sqrt[n]{a} \text{ gir alltid mening når } n \text{ er odde. } a \geq 0 \text{ når } n \text{ er partall.}}$

b) $e^y y' = e^{2x}$ (Separabel)

$$\int e^y y' dx = \int e^{2x} dx \qquad \boxed{\int e^x dx = e^x + C}$$

$$\int e^y dy = \int e^{2x} dx$$

$$e^y = \underbrace{\int e^{2x} dx}_{I}$$

$u = 2x$

$\frac{du}{dx} = 2$

$du = 2\,dx$

$\frac{du}{2} = dx$

$$I = \int e^u \frac{du}{2} = \int \frac{1}{2} e^u du$$

$$= \frac{1}{2} e^u + C$$

$$= \frac{1}{2} e^{2x} + C.$$

$$e^y = \frac{1}{2} e^{2x} + C$$

$$\ln(e^y) = \ln\left(\frac{1}{2}e^{2x} + C\right)$$

$$\underline{\underline{y = \ln\left(\frac{1}{2}e^{2x} + C\right)}} \qquad \left(\underbrace{\frac{1}{2}e^{2x}}_{>0} + C > 0\right)$$

$\boxed{\text{M.a.o. krever vi at } C \geq 0.}$

12.6 a) $y'' + 2y' - 5y = 0$ (lineær andre orders diff.likning m. konstante koeffisienter)

Karakteristisk polynom: $r^2 + 2r - 5 = 0$.

$$r = \frac{-2 \pm \sqrt{2^2 - 4(-5)}}{2} = \frac{-2 \pm \sqrt{4+20}}{2}$$

$$= \frac{-2 \pm \sqrt{24}}{2}$$

$24 = 4 \cdot 6$
$\sqrt{24} = \sqrt{4}\sqrt{6}$
$\sqrt{24} = 2\sqrt{6}$

$$= \frac{-2 \pm 2\sqrt{6}}{2}$$

$$= -1 \pm \sqrt{6}$$

To forskjellige reelle løsninger: $r_1 = -1 + \sqrt{6}$
$r_2 = -1 - \sqrt{6}$.

$$y = Ce^{r_1 x} + De^{r_2 x}$$

$$y = Ce^{(-1+\sqrt{6})x} + De^{(-1-\sqrt{6})x}$$

c) $y'' + 4y' + 4y = 0$

$r^2 + 4r + 4 = 0$

$$r = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 4}}{2} = \frac{-4}{2} = -2.$$

Én reell løsning: $y = Ce^{rx} + Dxe^{rx}$

$$y = Ce^{-2x} + Dxe^{-2x}$$

8 b) $3y'' + 6y' + 9y = 0$

$3r^2 + 6r + 9 = 0 \quad | \cdot \frac{1}{3}$

$r^2 + 2r + 3 = 0$

$$r = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 3}}{2}$$

$4 - 4 \cdot 3$
$= 4 - 12$
$= -8$

$$= \frac{-2 \pm i\sqrt{8}}{2}$$

$8 = 4 \cdot 2$
$\sqrt{8} = \sqrt{4} \cdot \sqrt{2}$
$\sqrt{8} = 2\sqrt{2}$

$$= \frac{-2 \pm i2\sqrt{2}}{2}$$

$$= -1 \pm i\sqrt{2}$$

$r_1 = -1 + i\sqrt{2}$, $r_2 = -1 - i\sqrt{2}$.

$r_1 = a + ib \quad | \; a = -1, \; b = \sqrt{2}$

$$y = e^{ax}(C\cos(bx) + D\sin(bx))$$

$$y = e^{-x}(C\cos(\sqrt{2}x) + D\sin(\sqrt{2}x))$$

12.9 Løs initialverdiproblemet:

b) $y' + xy = x$, (*) $y(0) = 0$.

Metode for første ordens difflikning

$y' + f(x)y = g(x)$

1) Finn en antiderivert $F(x) = \int f(x)\,dx$

2) Gang med $e^{F(x)}$ på begge sider.

3) Trekk sammen uttrykk på venstre

4) Integrer begge sider

$f(x) = x$, $g(x) = x$ i vår likning

1) $F(x) = \int x\,dx = \frac{1}{2}x^2$

2) $e^{\frac{1}{2}x^2}(y' + xy) = e^{\frac{1}{2}x^2} \cdot x$

$e^{\frac{1}{2}x^2}y' + e^{\frac{1}{2}x^2}xy = e^{\frac{1}{2}x^2} \cdot x$

3) $(e^{\frac{1}{2}x^2}y)' = e^{\frac{1}{2}x^2}x$.

4) $\int (e^{\frac{1}{2}x^2}y)'\,dx = \int e^{\frac{1}{2}x^2}x\,dx$.

(*) $e^{\frac{1}{2}x^2}y = \underbrace{\int e^{\frac{1}{2}x^2}x\,dx}_{I}$

$u = \frac{1}{2}x^2$

$\frac{du}{dx} = x$

$du = x\,dx$

$\frac{du}{x} = dx$

$I = \int e^u x \frac{du}{x} = \int e^u\,du = e^u + C$

$I = e^{\frac{1}{2}x^2} + C$

(*) $e^{\frac{1}{2}x^2}y = e^{\frac{1}{2}x^2} + C$ | Ganger med $e^{-\frac{1}{2}x^2} = \frac{1}{e^{\frac{1}{2}x^2}}$)

$y = e^{-\frac{1}{2}x^2}(e^{\frac{1}{2}x^2} + C)$

$y = 1 + e^{-\frac{1}{2}x^2}C$.

$y = 1 + Ce^{-\frac{1}{2}x^2}$

$y(0)=0$ | $y(0) = 1 + Ce^{-\frac{1}{2}\cdot 0^2} = 1 + Ce^0$

$= 1 + C$

$1 + C = 0 \Rightarrow C = -1$.

Partikulær løsning: $y = 1 - e^{-\frac{1}{2}x^2}$

c) $y' + 2y = x^2$, $y(1) = 2$

$e^{2x}y' + e^{2x}2y = e^{2x}x^2$. $(y' + f(x)y = g(x))$

$(e^{2x}y)' = e^{2x}x^2$. $(f(x) = 2,\ g(x) = x^2)$

$e^{2x}y = \underbrace{\int e^{2x} x^2\,dx}_{I}$ $(F(x) = 2x,\ e^{2x})$

$\left(\int e^{2x}dx = \frac{1}{2}e^{2x}\right)$

Delvis integrasjon: $I = \frac{1}{2}e^{2x}x^2 - \int \frac{1}{2}e^{2x}(2x)\,dx$

$I = \frac{1}{2}e^{2x}x^2 - \underbrace{\int e^{2x}x\,dx}_{I_1}$

$I_1 = \int e^{2x}x\,dx = \frac{1}{2}e^{2x}x - \int \frac{1}{2}e^{2x}\cdot 1\,dx$

$= \frac{1}{2}e^{2x}x - \frac{1}{2}\int e^{2x}\,dx$

$= \frac{1}{2}e^{2x}x - \frac{1}{2}\left(\frac{1}{2}e^{2x}\right) + C$

$= \frac{1}{2}e^{2x}x - \frac{1}{4}e^{2x} + C$.

$I = \frac{1}{2}e^{2x}x^2 - \left(\frac{1}{2}e^{2x}x - \frac{1}{4}e^{2x} + C\right)$

$I = \frac{1}{2}e^{2x}x^2 - \frac{1}{2}e^{2x}x + \frac{1}{4}e^{2x} - C$

$e^{2x}y = \frac{1}{2}e^{2x}x^2 - \frac{1}{2}e^{2x}x + \frac{1}{4}e^{2x} - C$ | e^{-2x}

$y = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4} - e^{-2x}C$. $\boxed{y(1) = 2}$

$y(1) = \cancel{\frac{1}{2} - \frac{1}{2}} + \frac{1}{4} - e^{-2\cdot 2}C$

$= \frac{1}{4} - e^{-4}C$

$\frac{1}{4} - e^{-4}C = 2$

$\frac{1}{4} = 2 + e^{-4}C$

$\frac{1}{4} - 2 = e^{-4}C$

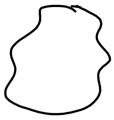
$\frac{1}{4} - \frac{8}{4} = e^{-4}C$

$\frac{-7}{4} = e^{-4}C$ | $\cdot e^4$

$\frac{-7}{4}e^4 = C \Rightarrow y(x) = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4} + \frac{7}{4}e^4 \cdot e^{-2x}$

$y(x) = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4} + \frac{7}{4}e^{4-2x}$

12.12 Har innsjø med laks



$L(t)$ = antall laks i sjøen ved tiden t

(t målt i uker)

Ved tiden $t=0$, så ut settes laksen for en sykdom.

$$L'(t) = -k\sqrt{L(t)}, \quad k>0.$$

Anta: $L(0)=1000$. $L(4)=720$.

Hvor mange uker tar det før det er tomt for laks?
Skal finne t s.a. $L(t)=0$.

Løsning: $L' = -k\sqrt{L}$ (variabelen er t)

$$\frac{L'}{\sqrt{L}} = -k \qquad \text{(deler på } \sqrt{L}\text{)}.$$

$$\int \frac{L'}{\sqrt{L}}\,dt = \int -k\,dt \qquad \text{(integrerer)}$$

$$\left[\ L'\,dt = \frac{dL}{dt}\,dt = dL\ \right]$$

$$\underbrace{\int \frac{1}{\sqrt{L}}\,dL}_{} = \int -k\,dt$$

$$2\sqrt{L} = -kt + C \qquad \left[\ \int \frac{1}{\sqrt{y}}\,dy = 2\sqrt{y} + C\ \right]$$

Husk fra tidligere oppgv.

$$\sqrt{L} = \frac{-kt+C}{2}$$

$$L(t) = \left(\frac{-kt+C}{2}\right)^2. \qquad \boxed{\begin{array}{l} L(0)=1000 \\ L(4)=720. \end{array}}$$

$$L(0) = \left(\frac{-k\cdot 0 + C}{2}\right)^2 = \left(\frac{C}{2}\right)^2 = \frac{C^2}{4}.$$

$$\frac{C^2}{4} = 1000. \Rightarrow C^2 = 4\cdot 1000$$

$$\Rightarrow C = \sqrt{4}\sqrt{1000}$$

$$C = 2\sqrt{1000}$$

$$\left(\frac{-kt+C}{2}\right)^2 = \left(\frac{kt-C}{2}\right)^2$$

$$L(4) = \left(\frac{-k\cdot 4 + C}{2}\right)^2$$

$$\left(\frac{-4k+C}{2}\right)^2 = 720. \Rightarrow \left(\frac{-4k+C}{2} = \pm\sqrt{720} = \pm 6\sqrt{20}.\right)$$

$$-4k + C = \pm 2\cdot 6\sqrt{20} = \pm 12\sqrt{20}.$$

$$-4k = \pm 12\sqrt{20} - C$$

$$\boxed{\begin{array}{l} 720 = 20\cdot 36. \\ \sqrt{720} = \sqrt{20}\sqrt{36} \\ \quad = 6\sqrt{20}. \end{array}}$$

$$k = \frac{\pm 12\sqrt{20} - 2\sqrt{1000}}{-4} = 2.4 \text{ eller } 29.2.$$

$$L(t) = \left(\frac{-kt+C}{2}\right)^2. \qquad \boxed{\begin{array}{l} C = 2\sqrt{1000} \\ k = 2.4 \text{ eller } 29.2 \end{array}}$$

$L(t) = 0$

$$\Leftrightarrow -kt + C = 0.$$

$$-kt = -C \quad \Big| \cdot \frac{1}{-k}$$

$$t = \frac{-C}{-k} = \frac{C}{k}$$

Sette inn

To muligheter: $\underbrace{t_1 = 26.4}_{k=2.4}$, $\cancel{t_2 = 2.2}$ ($k=29.2$)

Umulig siden $L(4)=720$.

Det tar 26.4 uker før innsjøen er tom for laks.

12.10 a) $y' + \frac{x}{2} y = 2x$, $y(2) = 5.$

$$\boxed{\int \frac{x}{2} dx = \frac{1}{2}\left(\frac{1}{2}x^2\right) = \frac{1}{4}x^2}$$

Ganger med $e^{\frac{1}{4}x^2}$:

$$\left(e^{\frac{1}{4}x^2} y\right)' = e^{\frac{1}{4}x^2} \cdot 2x.$$

Integrerer:

$$e^{\frac{1}{4}x^2} y = \underbrace{\int e^{\frac{1}{4}x^2} \cdot 2x \, dx}_{I}.$$

$u = \frac{1}{4}x^2$

$\frac{du}{dx} = \frac{1}{2}x$

$\frac{du}{\frac{1}{2}x} = \frac{\frac{1}{2}x \, dx}{\frac{1}{2}x}$

$\frac{2}{x} du = dx$

$$I = \int e^u \cdot 2\cancel{x} \cdot \frac{2}{\cancel{x}} du$$

$$= \int e^u \cdot 4 \, du = e^u \cdot 4 + C = 4e^{\frac{1}{4}x^2} + C$$

Ganger med $e^{-\frac{1}{4}x^2}$:

$$y = e^{-\frac{1}{4}x^2}\left(4e^{\frac{1}{4}x^2} + C\right)$$

$$y = 4 + e^{-\frac{1}{4}x^2} \cdot C \quad \text{(Generell løsning)}$$

$y(2) = 5$

$$y(2) = 4 + e^{-\frac{1}{4}\cdot 2^2} \cdot C$$

$$5 = 4 + e^{-1} \cdot C$$

$$1 = e^{-1} \cdot C \quad \Big| \cdot e = e^1$$

$$e = C \Rightarrow y = 4 + e^{-\frac{1}{4}x^2} \cdot e^1$$

$$y = 4 + e^{\left(-\frac{1}{2}x^2 + 1\right)}$$