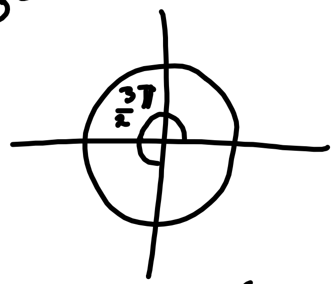
A7: 8abe, 10ab, 11ad, 12ad, 13ac
A8: 1, 2, 4

7.8: Regn om til radioner:

a) 270° b) -90° c) 540°

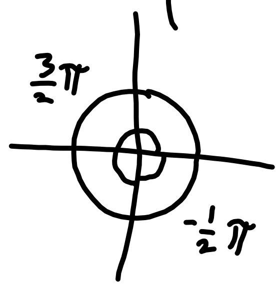
Formel: radioner $= \dfrac{\text{vinkel}}{360^\circ} \cdot 2\pi$.

a) $r = \underbrace{\left(\dfrac{270^\circ}{360^\circ}\right)}_{=\frac{3}{4}} \cdot 2\pi = \dfrac{3}{2}\pi$



Ligger i første omløp $\in [0, 2\pi)$. $(0 \leq v < 360^\circ)$

b) $r = \overbrace{\left(\dfrac{-90^\circ}{360^\circ}\right)}^{=-\frac{1}{4}} \cdot 2\pi = -\dfrac{1}{2}\pi$

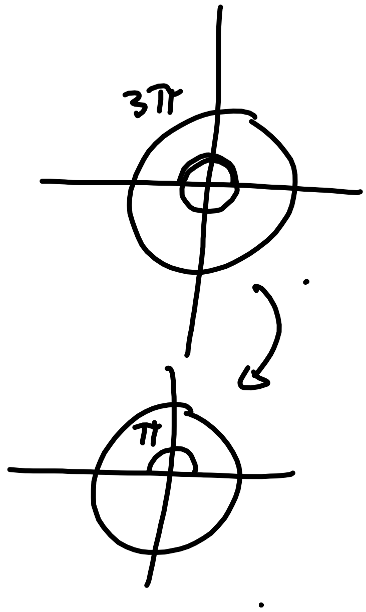


I første omløp:

Kan alltid legge til/trekke fra et multippel av 2π.

$-\dfrac{1}{2}\pi + 2\pi = \dfrac{3}{2}\pi$.

c) $r = \underbrace{\left(\dfrac{540^\circ}{360^\circ}\right)}_{=\frac{3}{2}} \cdot 2\pi = 3\pi$



$3\pi - 2\pi = \pi$

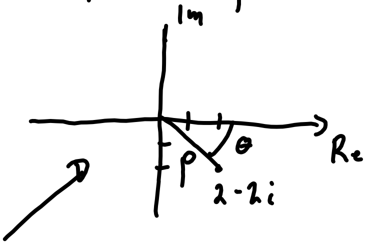
7.11 + 7.12) Skriv på polarform og eksponentialform.

a) $z = 2 - 2i$ b) $z = 2 + 2i$.

Polarform: $z = \rho\cos(\theta) + i\rho\sin(\theta)$

modulus

argumentet.



$$\boxed{\begin{aligned} \rho &= \sqrt{a^2+b^2} \\ z &= a + ib. \end{aligned}}$$

Eksponentialform: $z = \rho e^{i\theta}$.

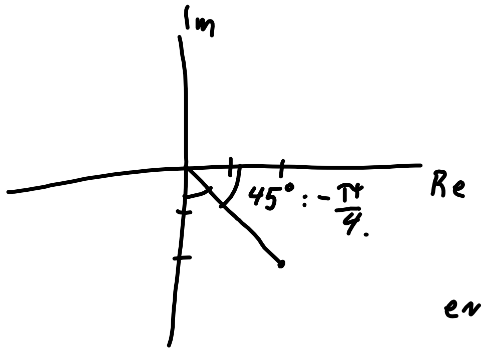
a) $z = \underset{a}{2} - \underset{b}{2i}$: modulus: $\rho = \sqrt{2^2 + (-2)^2}$

$= \sqrt{4+4}$

$= \sqrt{8}$

$= \sqrt{4 \cdot 2}$

$= \sqrt{4} \cdot \sqrt{2} = \underline{\underline{2\sqrt{2}}}$



$\theta = -\frac{\pi}{4}$

evt: $-\frac{\pi}{4} + 2\pi = \frac{7}{4}\pi$

$\underline{\underline{\theta = \frac{7\pi}{4}}}$

$z = 2\sqrt{2}\cos\left(\frac{7\pi}{4}\right) \underline{+ i2\sqrt{2}\sin\left(\frac{7\pi}{4}\right)}$.

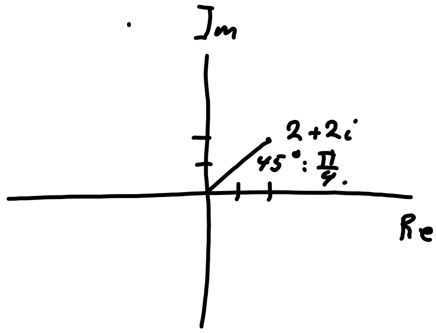
$\underline{\underline{z = 2\sqrt{2}\, e^{i\frac{7\pi}{4}}}}$.

b) $\underline{z = 2 + 2i}$:

$\rho = \sqrt{2^2 + 2^2}$

$= 2\sqrt{2}$ (se a)).

$\theta = \frac{\pi}{4}$.



$z = 2\sqrt{2}\cos\left(\frac{\pi}{4}\right) \underline{+ i2\sqrt{2}\sin\left(\frac{\pi}{4}\right)}$.

$\underline{\underline{z = 2\sqrt{2}\, e^{i\frac{\pi}{4}}}}$.

8.1 Løs differenslikningen

$$x_{n+2} - 5x_{n+1} + 4x_n = 0$$

m. initialbetingelse: $x_0 = 1$, $x_1 = -2$.

Andre ordens homogen differenslikning.

1) Karakteristisk likning:

$$x_{n+2} - 5x_{n+1} + 4x_n = 0$$
$$r^2 - 5r + 4 = 0$$

Løsn.: $r = \dfrac{5 \pm \sqrt{(-5)^2 - 4\cdot 4}}{2} = \dfrac{5 \pm \sqrt{25-16}}{2} = \dfrac{5 \pm \sqrt{9}}{2}$

$r = \dfrac{5 \pm 3}{2}$. $r_1 = 4$, $r_2 = 1$

Generell løsning: $x_n = C r_1^n + D r_2^n$.

$$x_n = C \cdot 4^n + D \cdot 1^n$$
$$x_n = C \cdot 4^n + D$$

$x_0 = 1$: $x_0 = C \cdot 4^0 + D = C + D$

$C + D = 1$.

$x_1 = -2$: $x_1 = C \cdot 4^1 + D = 4C + D$

$4C + D = -2$

1) $C + D = 1$
2) $4C + D = -2$

1) $D = 1 - C$

$\rightarrow$ 2) $4C + (1 - C) = -2$.

$$4C + 1 - C = -2$$
$$3C + 1 = -2$$
$$3C = -2 - 1$$
$$3C = -3$$
$$C = -1$$

1) $D = 1 - (-1) = 2$

$x_n = -1 \cdot 4^n + 2$ $\quad (C \cdot 4^n + D)$.

$x_n = -4^n + 2$

8.2 Løs $x_{n+2} - \frac{1}{2}x_{n+1} = \frac{1}{2}x_n$, $n \geq 0$.

a) m. initialbetingelser: $x_0 = 2$, $x_1 = \frac{1}{2}$.

Skriver om: $x_{n+2} - \frac{1}{2}x_{n+1} - \frac{1}{2}x_n = 0$

Homogen andre-ordens diff. likning.

$r^2 - \frac{1}{2}r - \frac{1}{2} = 0 \quad | \cdot 2$

$2r^2 - r - 1 = 0$.

$\leadsto r = \frac{1 \pm \sqrt{(-1)^2 - 4 \cdot 2 \cdot (-1)}}{2 \cdot 2} = \frac{1 \pm \sqrt{1+8}}{4} = \frac{1 \pm \sqrt{9}}{4} = \frac{1 \pm 3}{4}$.

$\underline{\underline{r_1 = 1}}$, $\underline{\underline{r_2 = -\frac{1}{2}}}$.

Generell formel: $x_n = Cr_1^n + Dr_2^n$.

$x_n = C \cdot 1 + D(-\frac{1}{2})^n$

$\underline{x_0 = 2}$: $x_0 = C + D(-\frac{1}{2})^0 = C + D$

$\underline{C + D = 2}$

$\underline{x_1 = \frac{1}{2}}$: $x_1 = C + D(-\frac{1}{2})^1 = C + D \cdot (-\frac{1}{2})$

$x_1 = C - \frac{1}{2}D$

$\underline{C - \frac{1}{2}D = \frac{1}{2}}$.

$$\begin{matrix} C + D = 2 \\ C - \frac{1}{2}D = \frac{1}{2} \end{matrix} \quad \Bigg| \quad \begin{bmatrix} 1 & 1 & 2 \\ 1 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \sim_{R_2 - R_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & \underline{-\frac{3}{2}} & -\frac{3}{2} \end{bmatrix}$$

$R_2 - R_1$:
$$\begin{array}{rrrr} R_2 & 1 & -\frac{1}{2} & \frac{1}{2} \\ -R_1 & -1 & -1 & -2 \\ \hline = & 0 & -\frac{3}{2} & -\frac{3}{2} \end{array}$$

$$\sim_{(-\frac{2}{3})} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \sim_{R_1 - R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

$\underline{C = 1}$

$\underline{D = 1}$

$x_n = C + D(-\frac{1}{2})^n$

$\underline{\underline{x_n = 1 + (-\frac{1}{2})^n}}$

b) Hva skjer når $n \to \infty$?

$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \left(1 + \underline{(-\frac{1}{2})^n}\right)$.

$a \in \mathbb{R}$ $\lim_{n\to\infty} a^n = 0$ hvis $|a| < 1$.

$\{a^n\}$ divergerer hvis $|a| > 1$.

$|-\frac{1}{2}| = \frac{1}{2} < 1. \Rightarrow \lim_{n\to\infty} (-\frac{1}{2})^n = 0$.

$\lim_{n\to\infty} \left(1 + (-\frac{1}{2})^n\right) = 1 + 0 = \underline{\underline{1}}$

Følgen $\{x_n\}_{n=0}^{\infty}$ konvergerer mot 1.

8.4 Løs: a) $x_{n+2} + x_{n+1} - 6x_n = 0,\ x_0 = 1,\ x_1 = -2.$

$r^2 + r - 6 = 0 \rightsquigarrow r = \frac{-1 \pm \sqrt{1^2 - 4(-6)}}{2}$

$= \frac{-1 \pm \sqrt{1+24}}{2}$

$= \frac{-1 \pm \sqrt{25}}{2}$

$= \frac{-1 \pm 5}{2}$. $r_1 = 2$, $r_2 = -3$

G. formel: $x_n = Cr_1^n + Dr_2^n$.

$x_n = C \cdot 2^n + D(-3)^n$.

$x_0 = 1$: $x_0 = C \cdot 2^0 + D(-3)^0$

$= C + D$

1) $C + D = 1$

$x_1 = -2$: $x_1 = C \cdot 2^1 + D(-3)^1$

$x_1 = 2C - 3D.$

2) $2C - 3D = -2.$

$L_1: C + D = 1$

$L_2: 2C - 3D = -2.$

$L_2 + 3L_1 = -2 + 3 \cdot 1 = 1.$

$L_2 + 3L_1 = (2C - 3D) + 3(C + D)$

$= 2C \cancel{-3D} + 3C \cancel{+3D}$

$= 5C.$

$5C = 1. \Longrightarrow C = \frac{1}{5}.$

$L_1: \frac{1}{5} + D = 1.$ $D = 1 - \frac{1}{5} = \frac{4}{5}$

$x_n = \frac{1}{5} \cdot 2^n + \frac{4}{5}(-3)^n.$

b) $x_{n+2} - x_n = 0$, $x_0 = 1$, $x_1 = 1$.

$r^2 - 1 = 0$ (koeffisienten foran $x_{n+1} = 0$).

$r^2 = 1 \Rightarrow r = \pm 1$. $r_1 = 1$, $r_2 = -1$.

Generelle formelen blir: $x_n = C \cdot 1^n + D \cdot (-1)^n$

$x_n = C + D(-1)^n$.

$x_0 = 1$: $x_0 = C + D(-1)^0 = C + D$.

L_1: $C + D = 1$

$x_1 = 1$: $x_1 = C + D(-1)^1 = C - D$

L_2: $C - D = 1$

L_2: $C = 1 + D$

L_1: $(1 + D) + D = 1$.

$2D + 1 = 1$.

$2D = 0$.

$D = 0 \Rightarrow$ L_2: $C = 1 + 0 = 1$

$$x_n = C + D(-1)^n$$
$$= 1 + 0 \cdot (-1)^n$$
$$x_n = 1$$

c) $x_{n+2} + 2x_{n+1} + 2x_n = 0, \quad x_0 = 1, \; x_1 = 2.$

$$r^2 + 2r + 2 = 0.$$

abc $\rightsquigarrow$ $r = \frac{-2 \pm \sqrt{2^2 - 4\cdot 2}}{2} = \frac{-2 \pm \sqrt{4-8}}{2}.$

$$r = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm i\sqrt{4}}{2}$$

$$= \frac{-2 \pm 2i}{2} = \underline{-1 \pm i.}$$

$\underline{r_1 = -1 + i}, \; \underline{r_2 = -1 - i}.$

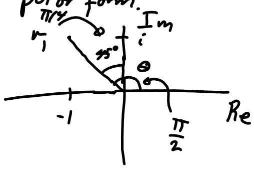
Røttene er kompleks konjugerte : $\overline{r_1} = r_2$.
Alltid tilfelle når vi starter med et reelt andregradspolynom.
Vi velger en av røttene, f.eks. $r_1 = -1 + i$.

Generelle formelen er

$x_n = C\rho^n \cos(n\theta) + D\rho^n \sin(n\theta). \qquad r_1 = \rho(\cos(\theta) + i\sin(\theta))$

Må skrive $r_1 = -1 + i$ på polar form.

$\underline{\theta = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}}$



$\rho = \sqrt{(-1)^2 + 1^2}$
$\underline{\rho = \sqrt{2}}$

$$x_n = C\rho^n \cos(n\theta) + D\rho^n \sin(n\theta).$$

$$x_n = C\sqrt{2}^n \cos\left(n\frac{3\pi}{4}\right) + D\sqrt{2}^n \sin\left(n\frac{3\pi}{4}\right).$$

$\underline{x_0 = 1}$: $\quad x_0 = C\sqrt{2}^0 \cos\left(0\cdot\frac{3\pi}{4}\right) + D\sqrt{2}^0 \sin\left(0\cdot\frac{3\pi}{4}\right).$

$$x_0 = C\cdot 1\cdot \underbrace{\cos(0)}_{=1} + \underbrace{D\cdot 1\cdot \sin(0)}_{=0}$$

$$x_0 = C.$$

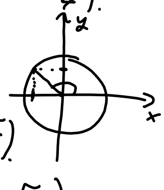
$$C = 1.$$

$\underline{x_1 = 2}$: $\quad x_1 = C\cdot\sqrt{2}^1 \cos\left(1\cdot\frac{3\pi}{4}\right) + D\sqrt{2}^1 \sin\left(1\cdot\frac{3\pi}{4}\right).$

$$x_1 = C\sqrt{2}\cos\left(\frac{3\pi}{4}\right) + D\sqrt{2}\sin\left(\frac{3\pi}{4}\right).$$

$$\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$$



$$x_1 = C\sqrt{2}\left(-\frac{\sqrt{2}}{2}\right) + D\sqrt{2}\left(\frac{\sqrt{2}}{2}\right)$$

$$x_1 = C\left(-\frac{\sqrt{2}\cdot\sqrt{2}}{2}\right) + D\left(\frac{\sqrt{2}\cdot\sqrt{2}}{2}\right)$$

$$x_1 = C\left(-\frac{2}{2}\right) + D\left(\frac{2}{2}\right)$$

$$x_1 = -C + D.$$

Vet at $C = 1$: $\quad x_1 = -1 + D.$

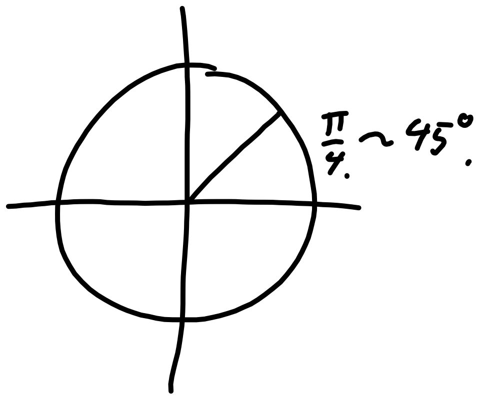
$$-1 + D = 2. \Rightarrow \underline{\underline{D = 3.}}$$

$$\underline{\underline{x_n = \sqrt{2}^n \cos\left(n\frac{3\pi}{4}\right) + 3\sqrt{2}^n \sin\left(n\frac{3\pi}{4}\right).}}$$

<u>7.10</u> Regn om til grader

a) $\frac{\pi}{4}$ b) $-\frac{\pi}{3}$.

a)



$$\text{vinkel} = \frac{\text{radianer}}{2\pi} \cdot 360^\circ.$$

$$V = \frac{\frac{\pi}{4}}{2\pi} \cdot 360^\circ$$

$$= \frac{\cancel{\pi}}{4 \cdot 2\cancel{\pi}} \cdot 360^\circ$$

$$= \frac{360^\circ}{8} = 45^\circ.$$

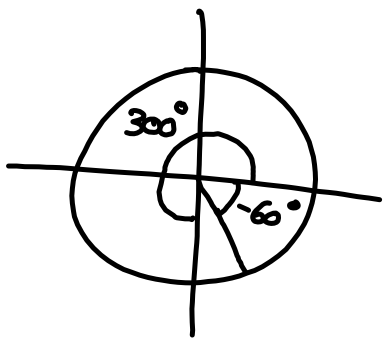
b) $-\frac{\pi}{3}$:

$$V = \frac{-\frac{\pi}{3}}{2\pi} \cdot 360^\circ$$

$$= \frac{-\cancel{\pi}}{3 \cdot 2\cancel{\pi}} \cdot 360^\circ$$

$$= -\frac{1}{6} \cdot 360^\circ$$

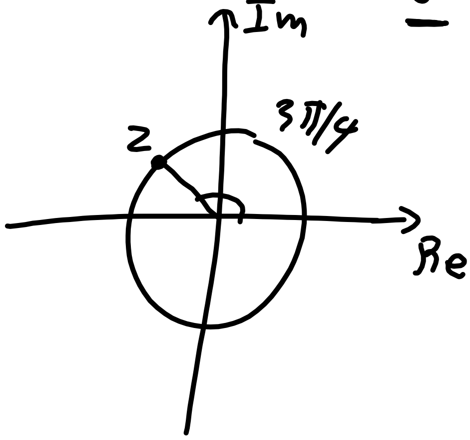
$$= -60^\circ.$$



$$-60^\circ + 360^\circ = 300^\circ.$$

13 Skriv på formen $a+ib$.

a) $\underbrace{z = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)}_{\text{polar form}}$ c.) $z = 2\left(\cos\left(\frac{\pi}{3}\right) - i\sin\left(\frac{\pi}{6}\right)\right)$.

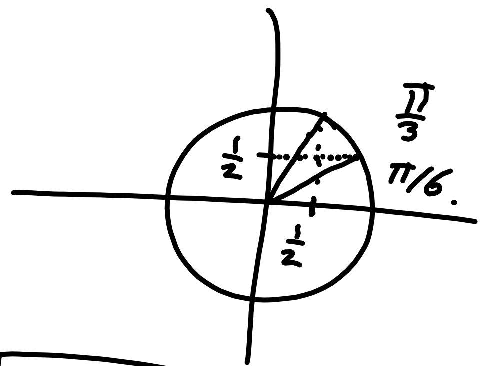


a) $\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

$z = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$. (har modulus 1).

c) $z = 2\left(\cos\left(\frac{\pi}{3}\right) - i\sin\left(\frac{\pi}{6}\right)\right)$..



$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$.

$z = 2\left(\frac{1}{2} - i\,\frac{1}{2}\right)$

$\underline{\underline{z = 1 - i}}$

Merk:

skal være $\frac{\pi}{3}$, ikke $\pi/6$...

ender opp med $\underline{\underline{z = 1 - i\sqrt{3}}}$