2014 : 1,3
2016 : 3
2015 : 3

2014 : 7 : Et likningssystem:

$$\begin{cases} x+y+z=1 \\ x-y+2z=0 \\ 2x+\alpha y+3z=\alpha+1 \end{cases}$$

$\alpha \in \mathbb{R}$., en konstant.

Finn de α s.a. systemet har ∞ mange løsninger. Finn løsningene i så fall.

Matriseformen til likningen er: $Ax=b$

$$A=\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & \alpha & 3 \end{bmatrix}, \quad b=\begin{bmatrix} 1 \\ 0 \\ \alpha+1 \end{bmatrix}.$$

$b \neq 0$ M.a.o. Likningen er inhomogen $(\forall \alpha)$

$\boxed{\text{Har vendelig mange løsninger} \iff \det(A)=0.}$

Må regne ut $\det(A)$.

$$\det\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & \alpha & 3 \end{bmatrix} = 1\cdot\begin{vmatrix} -1 & 2 \\ \alpha & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + 1\cdot\begin{vmatrix} 1 & -1 \\ 2 & \alpha \end{vmatrix}$$

$$=(-1\cdot 3-2\alpha)-(1\cdot 3-2\cdot 2)+(1\cdot\alpha-(-1)\cdot 2)$$

$$=(-3-2\alpha)-(\underbrace{3-4}_{=-1})+(\alpha+2)$$

$$=-3-2\alpha+1+\alpha+2$$

$$=0-\alpha=-\alpha.$$

$\det(A)=-\alpha$. $\det(A)=0 \iff \alpha=0$.

$$\left.\begin{matrix} x+y+z=1 \\ x-y+2z=0 \\ 2x \quad +3z=1 \end{matrix}\right\} \text{ Har utvidet matrise } [A \mid b]$$

$$\begin{array}{c} \\ I_2-I_1 \\ \\ \end{array}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & 0 \\ 2 & 0 & 3 & 1 \end{array}\right] \sim \begin{array}{c} \\ \\ I_3-2I_1 \end{array}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -1 \\ 2 & 0 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -1 \\ 0 & -2 & 1 & -1 \end{array}\right]$$

like

$$\begin{array}{cccc} 1 & -1 & 2 & 0 \\ -1 & -1 & -1 & -1 \\ \hline 0 & -2 & 1 & -1 \end{array} \quad \begin{array}{cccc} 2 & 0 & 3 & 1 \\ -2 & -2 & -2 & -2 \\ \hline 0 & -2 & 1 & -1 \end{array} \quad \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Redusert form

L_1: $x+y+z=1$
L_2: $-2y+z=-1$

z en frivariabel. $\rightsquigarrow$ Parameter $t=z$.

L_2: $-2y+t=-1$
$-2y=-1-t$
$2y=1+t$
$y=\frac{1}{2}+\frac{1}{2}t$

L_1: $x+\left(\frac{1}{2}+\frac{1}{2}t\right)+t=1$
$x+\frac{1}{2}+\frac{1}{2}t+t=1$
$x+\frac{1}{2}+\frac{3}{2}t=1$
$x=1-\frac{1}{2}-\frac{3}{2}t$
$x=\frac{1}{2}-\frac{3}{2}t$

$x=\frac{1}{2}-\frac{3}{2}t$, $y=\frac{1}{2}+\frac{1}{2}t$, $z=t$ er løsningene for $t \in \mathbb{R}$.

Løsningsmengden: $\left\{\left(\frac{1}{2}-\frac{3}{2}t, \frac{1}{2}+\frac{1}{2}t, t\right) : t \in \mathbb{R}\right\}$.

2014, 3: En funksjon tilfredsstiller

$$f'(x) = e^x \sin(x), \qquad f(0) = 1.$$

Finn $f(x)$.

$f(x)$ er en anti-derivert til $e^x \sin(x)$.

Integrerer $e^x \sin(x)$:

$$I = \int e^x \sin(x)\,dx = e^x \sin(x) - \underbrace{\int e^x \cos(x)\,dx}_{I_1}.$$

$$I_1 = \int e^x \cos(x)\,dx = e^x \cos(x) - \int e^x(-\sin(x))\,dx$$

$$= e^x \cos(x) + \underbrace{\int e^x \sin(x)\,dx}_{I}$$

$$I = e^x \sin(x) - I_1$$
$$I_1 = e^x \cos(x) + I.$$

$$I = e^x \sin(x) - (e^x \cos(x) + I)$$
$$I = e^x \sin(x) - e^x \cos(x) - I$$
$$2I = e^x(\sin(x) - \cos(x))$$
$$I = \frac{e^x}{2}(\sin(x) - \cos(x)) + C$$

$$f(x) = \frac{e^x}{2}(\sin(x) - \cos(x)) + C, \qquad f(0) = 1.$$

$$\frac{e^0}{2}(\sin(0) - \cos(0)) + C = 1$$
$$\frac{1}{2}(0 - 1) + C = 1$$
$$-\frac{1}{2} + C = 1$$
$$C = \frac{3}{2}.$$

$$f(x) = \frac{e^x}{2}(\sin(x) - \cos(x)) + \frac{3}{2}$$

2016,3: En første-ordens difflikning
$$xy' + 2y = -1 \quad , \; x > 0.$$

a) Finn alle konstante løsninger.

$y(x) = C$, C konstant tall.

$y' = 0$. Setter inn: $x \cdot 0 + 2C = -1$
$$2C = -1$$
$$C = -\frac{1}{2}$$

$y = -\frac{1}{2}$ er den eneste konstante løsningen.

b) Løs difflikningen, gitt initialverdien $y(1) = 0$.
$$xy' + 2y = -1. \quad \left(y' + f(x)y = g(x) \right)$$

Deler på x: $y' + \frac{2}{x}y = -\frac{1}{x}$ (siden $x > 0$)

Integrerende faktor: $e^{\int f(x)dx} = e^{\int \frac{2}{x}dx}$
$$\int \frac{2}{x}dx = 2\int \frac{1}{x}dx = 2\ln(x). \quad (x > 0)$$
$$e^{2\ln(x)} = (e^{\ln(x)})^2 = x^2 \qquad \left(e^{\ln(x)} = x \right)$$

Integrerende faktor: x^2. Likningen blir da:
$$(yx^2)' = x^2 \cdot \left(-\frac{1}{x}\right)$$
$$(yx^2)' = -x.$$

Integrerer begge sider:
$$yx^2 = \int -x\,dx \qquad \text{regel:} \left[\int x^n dx = \frac{x^{n+1}}{n+1} \right]$$
$$\frac{yx^2}{x^2} = \frac{-\frac{1}{2}x^2 + C}{x^2}$$
$$y = -\frac{1}{2} + \frac{C}{x^2}. \quad \text{Den generelle løsningen.}$$

$y(1) = 0$:
$$\boxed{\begin{aligned} -\frac{1}{2} + \frac{C}{1^2} &= 0 \\ -\frac{1}{2} + C &= 0 \\ C &= \frac{1}{2}. \end{aligned}}$$

$$y = -\frac{1}{2} + \frac{1}{2x^2}$$

Den spesielle løsningen.

$$y' + \frac{2}{x}y = -\frac{1}{x}$$

x^2. $\qquad (ye^{\int f(x)dx})'$
$$x^2\left(y' + \frac{2}{x}y\right) = x^2\left(-\frac{1}{x}\right).$$
$$(yx^2)' = y' \cdot x^2 + y \cdot 2x$$
$$x^2y' + 2xy$$

2015, 3 (4):

a) Skriv $\sin(\frac{1}{2}t)+\sqrt{3}\cos(\frac{1}{2}t)$ på faseform $A\cos(\omega t-\varphi)$.

$\boxed{C\sin(bt)+D\cos(bt)} = A\cos(\omega t-\varphi)$ der:

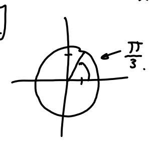
$A=\sqrt{C^2+D^2}$.

$\cos(\varphi)=\frac{C}{A}$, $\sin(\varphi)=\frac{D}{A}$. $(\omega=b)$

$\sin(\frac{1}{2}t)+\sqrt{3}\cos(\frac{1}{2}t)$. $C=1$, $D=\sqrt{3}$, $b=\frac{1}{2}$.

$A=\sqrt{1^2+\sqrt{3}^2}=\sqrt{1+3}=2$. $\boxed{A=2}$

(φ): $\cos(\varphi)=\frac{1}{2}$, $\sin(\varphi)=\frac{\sqrt{3}}{2}$.



$\boxed{\varphi=\frac{\pi}{3}}$ $\boxed{\omega=\frac{1}{2}}$

Faseformen: $A\cos(\omega t-\varphi)=\underline{\underline{2\cos(\frac{1}{2}t-\frac{\pi}{3})}}$

b) Løs difflikningen

$4y''+y=0$. $y(\pi)=1$, $y'(\pi)=-\frac{\sqrt{3}}{2}$.

Andre-ordens lineær $\underset{\text{homogen}}{\wedge}$ difflikning med konstante koeff.

Karakteristisk likning: $4r^2+1=0$.

$\frac{4r^2}{4}=\frac{-1}{4}$

$r^2=-\frac{1}{4}$.

$r=\pm\sqrt{-\frac{1}{4}}$

$r=\pm i\sqrt{\frac{1}{4}}=\pm i\frac{1}{\sqrt{4}}=\pm i\cdot\frac{1}{2}$.

Velger én rot $r_1=\frac{1}{2}i$.

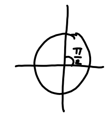
Generelt: hvis $r_1=a+ib$, da er den generelle løsningen: $y(t)=Ce^{at}\cos(bt)+De^{at}\sin(bt)$.

$r_1=\frac{1}{2}i=\underline{0}+i\cdot\underline{\frac{1}{2}}$. $a=0$, $b=\frac{1}{2}$.

$y=\underbrace{C\cos(\frac{1}{2}t)+D\sin(\frac{1}{2}t)}_{\text{Generell løsning}}$ $\left(e^{0\cdot t}=1\right)$.

$\underline{y(\pi)=1}$, $\boxed{y'(\pi)=-\frac{\sqrt{3}}{2}}$

$\underline{y(\pi)=1}$: $C\cos(\frac{\pi}{2})+D\sin(\frac{\pi}{2})=1$.



$C\cdot 0 + D\cdot 1 = 1$

$\underline{D = 1}$.

$y=C\cos(\frac{1}{2}t)+\sin(\frac{1}{2}t)$.

$y'=C(-\frac{1}{2}\sin(\frac{1}{2}t))+\frac{1}{2}\cos(\frac{1}{2}t)$

$y'=-\frac{C}{2}\sin(\frac{1}{2}t)+\frac{1}{2}\cos(\frac{1}{2}t)$.

Kjerneregel:
$u=\frac{1}{2}t$. $u'=\frac{1}{2}$.
$\cos(u)'=u'\cdot(-\sin(u))$
$=-\frac{1}{2}\sin(u)$
$=-\frac{1}{2}\sin(\frac{1}{2}t)$.

$y'(\pi)=-\frac{\sqrt{3}}{2}$

den deriverte.

$-\frac{C}{2}\underbrace{\sin(\frac{\pi}{2})}_{=1}+\frac{1}{2}\underbrace{\cos(\frac{\pi}{2})}_{=0}=-\frac{\sqrt{3}}{2}$.

$-\frac{C}{2}\cdot 1+\frac{1}{2}\cdot 0=-\frac{\sqrt{3}}{2}$.

$-\frac{C}{2}=-\frac{\sqrt{3}}{2}$

$\frac{C}{2}=\frac{\sqrt{3}}{2}$ $\Big|\cdot 2$

$C=\sqrt{3}$

$D=1$, $C=\sqrt{3}$

$\underline{\underline{y(t)=\sqrt{3}\cos(\frac{1}{2}t)+\sin(\frac{1}{2}t)}}$

Den partikulære løsningen

Finn en antiderivert til funksjonen f(x) = x · e ax2 , der a er et reellt tall, ulikt 0.

2015, 4 :

Finn en antiderivert til $f(x) = xe^{ax^2}$, der $a \neq 0$.

$$I = \int x e^{ax^2} dx.$$

Substitusjon:

$u = ax^2$.

$\frac{du}{dx} = 2ax$

$du = 2ax\,dx$

$\frac{du}{2ax} = dx$

$$I = \int x e^u \frac{du}{2ax} = \int \frac{\cancel{x} e^u}{2a\cancel{x}} du = \int \frac{e^u}{2a} du$$

$$I = \frac{1}{2a} e^u + C$$

$$I = \frac{1}{2a} e^{ax^2} + C.$$

Velger $C = 0$, får vi en antiderivert $F(x) = \frac{1}{2a} e^{ax^2}$

$$F(x) = \frac{1}{2a} e^{ax^2}$$

b) Første-ordens difflikning

$$y' + xy = -x \quad , \quad y(1) = 0$$

~~Fin~~ Løs likningen, og undersøk hva som skjer når $x \to \infty$. Dvs $\lim_{x\to\infty} y(x)$. c)

$$y' + xy = -x \qquad \left(\begin{array}{l}\text{Standard form} \\ y' + f(x)y = g(x)\end{array}\right)$$

Integrerende faktor: $e^{\int x\,dx} = e^{\frac{1}{2}x^2}$

$\Rightarrow$ likningen blir: (ganger med den integrerende faktor)

$$\underline{\left(e^{\frac{1}{2}x^2}y\right)'} = e^{\frac{1}{2}x^2}(-x) = -e^{\frac{1}{2}x^2}x = -xe^{\frac{1}{2}x^2}.$$

Integrerer begge sider: $\left(e^{\frac{1}{2}x^2}y\right)' = -xe^{\frac{1}{2}x^2}$

$$e^{\frac{1}{2}x^2}y = \int -xe^{\frac{1}{2}x^2}dx.$$

$$e^{\frac{1}{2}x^2}y = -\int xe^{\frac{1}{2}x^2}dx.$$

$$\boxed{\text{En antiderivert til } xe^{ax^2} \text{ er } \frac{e^{ax^2}}{2a}. \quad a)}$$

I vårt tilfelle: $a = \frac{1}{2}$.

$$\int xe^{\frac{1}{2}x^2}dx = \frac{e^{\frac{1}{2}x^2}}{2\cdot\frac{1}{2}} + C = e^{\frac{1}{2}x^2} + C$$

$$\Rightarrow e^{\frac{1}{2}x^2}y = -\left(e^{\frac{1}{2}x^2} + C\right)$$

$$e^{\frac{1}{2}x^2}y = -e^{\frac{1}{2}x^2} - C \quad \Big| \cdot e^{-\frac{1}{2}x^2} = \frac{1}{e^{\frac{1}{2}x^2}}$$

$$y = \left(-e^{\frac{1}{2}x^2} - C\right)e^{-\frac{1}{2}x^2}$$

$$y = -e^{\frac{1}{2}x^2}\cdot e^{-\frac{1}{2}x^2} - Ce^{-\frac{1}{2}x^2}$$

$$\underline{y = -1 - Ce^{-\frac{1}{2}x^2}}.$$ Den generelle løsningen.

$$\underline{y = -1 - Ce^{-\frac{1}{2}x^2}} \qquad y(1) = 0.$$

$$-1 - Ce^{-\frac{1}{2}\cdot 1^2} = 0.$$

$$-1 - Ce^{-\frac{1}{2}} = 0$$

$$-1 = Ce^{-\frac{1}{2}} \quad \Big| \cdot e^{\frac{1}{2}}$$

$$-e^{\frac{1}{2}} = C$$

$$C = -e^{\frac{1}{2}} \Rightarrow y = -1 - (-e^{\frac{1}{2}})e^{-\frac{1}{2}x^2}$$

$$\underline{\underline{y = -1 + e^{\frac{1}{2}}e^{-\frac{1}{2}x^2}}}$$

Partikulær-løsning.