Mandatory assignment MAT4300 - Fall 2008

Deadline Friday November 7, 14.30

Where Expedition office of the Department of Mathematics, 7^{th} floor, Niels Henrik Abel's house.

All answers must be justified.

Problem 1

a) State the monotone convergence theorem and the Lebesgue dominated convergence theorem.
For each of the two theorems, find an example where the theorem chosen applies, but not the other one.

b) Let $X = [1, \infty)$, $\mathbf{X}$ be the Borel algebra on X and let λ be the Lebesgue measure on $\mathbf{X}$.
Let

$$f_n(x) = x^{-2}\cos(\frac{x}{n})$$

for $n \in \mathbb{N}$.
Calculate

$$\lim_{n\to\infty} \int f_n d\lambda.$$

c) Let

$$g_n(x) = x^{-\frac{1}{3}}\cos(\frac{x^2}{n})$$

for each $n \in \mathbb{N}$.
For which p, $1 \leq p \leq \infty$, will $g_n \in L_p$ for all n?
Will $\{g_n\}_{n\in\mathbb{N}}$ converge in L_p for these p?
If this is the case, find the limit in L_p of the sequence.

d) Let $\{c_m\}_{m\in\mathbb{N}}$ be an arbitrary sequence in X and let

$h_n(x) = 1$ if

$$c_m - 2^{-(m+n)} < x < c_m + 2^{-(n+m)}$$

for some $m \in \mathbb{N}$,

$h_n(x) = f_n(x)$ otherwise.

Calculate $\lim_{n\to\infty} \int h_n d\lambda$.

Problem 2 Let $X = [0,1]$, $\mathbf{X}$ the Borel algebra on X and λ the Lebesgue measure on $\mathbf{X}$.
We will define $f_n : X \to \{0,1\}$ for each $n \in \mathbb{N}$ as follows:
Let m be such that

$$m^2 \leq n < (m+1)^2$$

and let k be such that $n = m^2 + k$.
Let

$f_n(x) = 1$ if

$$\frac{k}{2m+1} \leq x \leq \frac{k+1}{2m+1}.$$

$f_n(x) = 0$ otherwise.

a) Show that $\{f_n(x)\}_{n\in\mathbb{N}}$ does not have a limit for any $x \in X$.

b) Let $1 \leq p < \infty$. Show that $\{f_n\}_{n\in\mathbb{N}}$ has a limit in L_p and find this limit.

c) Does $\{f_n\}_{n\in\mathbb{N}}$ have a limit in L_∞?

Problem 3 Let $X = \{1, \ldots, n\}$ for some n, $\mathbf{X}$ be the powerset of X and μ the counting measure on $\mathbf{X}$.

a) Reformulate Hölder's inequality for $(X, \mathbf{X}, \mu)$ to a statement about ordered n-tuples of reals.

b) For every $f : X \to \mathbb{R}$, show that $f \in L_p$ for all p with $1 \leq p \leq \infty$ and show that

$$||f]]_\infty = \lim_{p\to\infty} ||f||_p.$$

Problem 4 A toss of a coin can be modeled as the random selection of a number in $\{0,1\}$ with equal probability. Let $P = \{0,1\}$ with the probability measure determined by $\lambda(\{0\}) = \lambda(\{1\}) = \frac{1}{2}$.
In order to investigate tosses of coins in the long run, one way is to form the set X of infinite sequences from P. We will view an infinite sequence as a function $\alpha : \mathbb{N} \to \{0,1\}$.
If $a_1, \ldots, a_n$ is a finite sequence from P, we let $B_{a_1 \cdots a_n}$ be the set of $\alpha \in X$ such that $\alpha(i) = a_i$ for all $i \leq n$.
We let $\mathbf{X}$ be the σ-algebra on X generated from the sets $B_{a_1 \cdots a_n}$ and we let μ be a measure on $\mathbf{X}$ such that $\mu(B_{a_1 \cdots a_n}) = 2^{-n}$ for all n and finite sequences $a_1, \ldots, a_n$ from P.
(You may assume the existence of μ without proof)

a) Let $K \subseteq \mathbb{N}$ be a finite set with k elements, and let $\beta : K \to P$.
Let B_β be the set of $\alpha \in X$ that extends β.
Show that $B_\beta \in \mathbf{X}$ and that $\mu(B_\beta) = 2^{-k}$.

b) Show that the set Y of sequences $\alpha \in X$ such that α contains infinitely many segments with ten zeroes in a row,

$$\ldots, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \ldots$$

is in $\mathbf{X}$ and that $\mu(Y) = 1$.

c) Let $\alpha \in X$. For $n \in \mathbb{N}$, let 0_n be the number of $i \leq n$ such that $\alpha(i) = 0$. Show that

$$Z = \{\alpha \in X \mid \lim_{n \to \infty} \frac{0_n(\alpha)}{n} = \frac{1}{2}\} \in \mathbf{X}.$$

d) (Optional) Show that $\mu(Z) = 1$, where Z is as in c).

Problem 5 Let $(X, \mathbf{X})$ be a measurable space and let $\{\mu_n\}_{n \in \mathbb{N}}$ be a sequence of measures on $\mathbf{X}$.
Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of positive reals and let

$$\mu(E) = \sum_{n=1}^{\infty} a_n \mu_n(E)$$

for $E \in \mathbf{X}$.

a) Show that μ is a measure and show that μ is a finite measure exactly when

$$\sum_{n=1}^{\infty} a_n \mu_n(X) < \infty.$$

b) Let λ be another measure on $\mathbf{X}$. Show that $\mu \ll \lambda$ if and only if $\mu_n \ll \lambda$ for all n, and that λ and μ are mutually singular, $\lambda \perp \mu$, if and only if $\lambda \perp \mu_n$ for each n.

You may choose between writing in English or in Norwegian. Mixtures of the two will also be accepted.
Other Scandinavian languages are acceptable by the general rules of UiO.

A miniature dictionary

- counting measure - tellemål
- expedition office - ekspedisjonen
- integrable function - integrerbar funksjon
- justified - begrunnet
- mandatory assignment - obligatorisk oppgave
- measure - mål
- measure space - målrom
- measurable space - målbart rom
- mutually singular - gjensidig singulære
- optional - frivillig
- power set - potensmengde
- simple function - enkel funksjon
- singular - singulær
- σ-finite - σ-endelig