MAT3300/4300 - Fall 09 - Extra-exercise 2

As in Extra-Exercise 1, we let X be a nonempty set, $f : X \to [0, \infty]$ and μ_f be the measure on $\mathcal{P}(X)$ given by

$$\mu_f(A) = \sum_{x \in A} f(x)\,, \quad A \in \mathcal{P}(X)\,.$$

a) Show that μ_f is σ-finite if and only if $f(x) < \infty$ for all $x \in X$ and $\{x \in X \mid f(x) \neq 0\}$ is countable.

b) Set $X = [0, 1]$ and define $f : X \to [0, \infty)$ by $f(x) = 1/n$ when $x = m/n \in \mathbb{Q} \cap [0, 1]$, while $f(x) = 0$ when $x \in [0, 1] \setminus \mathbb{Q}$.
(Here, we always use the standard representation of a rational number x given by $x = m/n$ where $m \in \mathbb{Z}, n \in \mathbb{N}$ and m, n are relatively prime, i.e. 1 is their only common divisor in $\mathbb{N}$).

Check that μ_f is σ-finite, but not finite.