MAT3300/4300 - Fall 09 - Extra-exercises 3, 4 and 5

Unless otherwise specified, $(X, \mathcal{A}, \mu)$ denotes a measure space, and
$\mathcal{M} = \mathcal{M}(\mathcal{A}),\ \bar{\mathcal{M}} = \mathcal{M}_{\bar{\mathbb{R}}}(\mathcal{A}),\ \bar{\mathcal{L}}^1(\mu) = \mathcal{L}^1_{\bar{\mathbb{R}}}(\mu)$.

Extra-Exercise 3

Let $E \in \mathcal{A}$ and set $\mathcal{A}_E = \{A \cap E \mid A \in \mathcal{A}\} \subset \mathcal{A}$. Let $\mu_E : \mathcal{A}_E \to [0, \infty]$ be the restriction of μ to $\mathcal{A}_E$, i.e. μ_E is given by $\mu_E(B) = \mu(B)\,,\ B \in \mathcal{A}_E$.

As we have seen in Ch. 3 and 4, $\mathcal{A}_E$ is a σ-algebra in E and μ_E is measure on $\mathcal{A}_E$.

If f.ex. $X = \mathbb{R},\ \mathcal{A} = \mathcal{B}\ (= \mathcal{B}(\mathbb{R})),\ \mu = \lambda$ and $E = [a, b],\ \mathcal{B}_{[a,b]}$ is called the Borel σ-algebra in $[a, b]$ and $\lambda_{[a,b]}$ is the Lebesgue measure on $\mathcal{B}_{[a,b]}$.

We set $\mathcal{M}_E = \mathcal{M}(\mathcal{A}_E), \bar{\mathcal{M}}_E = \mathcal{M}_{\bar{\mathbb{R}}}(\mathcal{A}_E)$.

Let $f \in \mathcal{M}$ [or $\bar{\mathcal{M}}$] and define f_E by restriction, i.e. $f_E(x) = f(x), x \in E$.

a) Check that $f_E \in \mathcal{M}_E$ [or $\bar{\mathcal{M}}_E$].

b) Assume $f \geq 0$. Check that $\int_E f\, d\mu = \int f_E\, d\mu_E$ and, more generally,

$$\int_B f\, d\mu = \int_B f_E\, d\mu_E \quad \text{for all } B \in \mathcal{A}_E\,.$$

c) Assume $f \in \mathcal{L}^1(\mu)$ [or $\bar{\mathcal{L}}^1(\mu)$]. Check that $f_E \in \mathcal{L}^1(\mu_E)$ [or $\bar{\mathcal{L}}^1(\mu_E)$],

$$\int_E f\, d\mu = \int f_E\, d\mu_E\,, \text{ and} \int_B f\, d\mu = \int_B f_E\, d\mu_E \quad \text{for all } B \in \mathcal{A}_E\,.$$

d) Assume $g \in \mathcal{M}$ [or $\bar{\mathcal{M}}$] is such that $f = g$ μ-a.e.
Check that $f_E = g_E$ μ_E-a.e.

Let now $h \in \mathcal{M}_E$ [or $\bar{\mathcal{M}}_E$] and define
$\tilde{h}(x) = h(x)$ when $x \in E$, while $\tilde{h}(x) = 0$ when $x \in E^c$.

e) Check that $\tilde{h} \in \mathcal{M}$ [or $\bar{\mathcal{M}}$].

f) Assume $h \geq 0$, so $\tilde{h} \geq 0$. Check that $\int \tilde{h}\, d\mu = \int h\, d\mu_E$ and, more generally,

$$\int_A \tilde{h}\, d\mu = \int_{A\cap E} h\, d\mu_E \quad \text{for all } A \in \mathcal{A}\,.$$

g) Assume $h \in \mathcal{L}^1(\mu_E)$ [or $\bar{\mathcal{L}}^1(\mu_E)$]. Check that $\tilde{h} \in \mathcal{L}^1(\mu)$ [or $\bar{\mathcal{L}}^1(\mu)$],

$$\int \tilde{h}\, d\mu = \int h\, d\mu_E\,, \text{ and } \int_A \tilde{h}\, d\mu = \int_{A\cap E} h\, d\mu_E \quad \text{for all } A \in \mathcal{A}\,.$$

h) Assume $k \in \mathcal{M}_E$ [or $\bar{\mathcal{M}}_E$] is such that $h = k$ μ_E-a.e.
Check that $\tilde{h} = \tilde{k}$ μ-a.e.

Extra-Exercise 4.

Let $\{f_j\}_{j\in\mathbb{N}} \subset \mathcal{M}$ and set $E = \{x \in X \mid \lim_{j\to\infty} f_j(x) \text{ exists }\}$.

a) Show that $E \in \mathcal{A}$. [This is not obvious. It may be helpful to have in mind the Cauchy criterion for convergence of sequences].

b) Define $f : X \to \mathbb{R}$ by $f(x) = \lim_{j\to\infty} f_j(x)$ when $x \in E$, while $f(x) = 0$ when $x \in E^c$. Show that $f \in \mathcal{M}$.

Note : These statements also hold if we assume $\{f_j\}_{j\in\mathbb{N}} \subset \bar{\mathcal{M}}$. Further, if $\mu(E^c) = 0$, i.e. $\lim_{j\to\infty} f_j(x)$ exists μ-a.e., it is common to be somewhat sloppy and refer to f defined in b) by writing "let $f = \lim_{j\to\infty} f_j$".

Extra-Exercise 5.

A measure space $(X, \mathcal{A}, \mu)$ [or just μ] is called *complete* whenever $N \in \mathcal{A}$, $\mu(N) = 0$ and $B \subset N$ implies $B \in \mathcal{A}$ (in which case $\mu(B) = 0$).

In some aspects, complete spaces are nicer to work with, and we will give an example of this fact in this exercise.

Many measure spaces are not complete. For example, it is known that $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ is not complete. But, as we pointed out in the lectures, in the setting of Caratheodory's theorem (cf. T6.1), the measure $\bar{\mu}$ on $\mathcal{A}^*$ obtained by restricting the outer measure μ^* to $\mathcal{A}^*$ is easily seen to be complete. Especially, this means that we could have chosen to work with the complete (Lebesgue) measure $\bar{\lambda}$ on the σ-algebra $\mathcal{L}(\mathbb{R}) = \mathcal{A}^*$ (which is known to be bigger than $\mathcal{B}(\mathbb{R})$, and called the Lebesgue σ-algebra in $\mathbb{R}$).

There are several exercises throughout the book which deal with the so-called *completion* of a measure space. We sorrily don't have time to look at these, as they are somewhat technical and would require more time than we have in this course. But we recommend that students who are interested to learn more spend some hours on these.

a) Assume $(X, \mathcal{A}, \mu)$ is not complete. Pick $N \in \mathcal{A}$ and $B \subset N$ such that $\mu(N) = 0$ and $B \notin \mathcal{A}$.

Set $C = N \setminus B$ and $f = \mathbf{1}_B + 2 \cdot \mathbf{1}_C$. Check that $f = 0$ μ-a.e., but $f \notin \mathcal{M}$.

Since $0 \in \mathcal{M}$, this means that we can have $g \in \mathcal{M},\ f = g$ μ-a.e. and $f \notin \mathcal{M}$.

b) Assume $(X, \mathcal{A}, \mu)$ is complete, $f : X \to \mathbb{R}$, $g \in \mathcal{M}$, and $f = g$ μ-a.e. Show that $f \in \mathcal{M}$.