MAT3300/4300 - Fall 09 - Extra-exercise 1

Let X be a nonempty set and let $f : X \to [0, \infty]$.

Define $\mu_f : \mathcal{P}(X) \to [0, \infty]$ by

$$\mu_f(A) = \sum_{x \in A} f(x)\,, \quad A \in \mathcal{P}(X)\,.$$

a) Show that μ_f is measure on $\mathcal{P}(X)$.

b) Set $X = \mathbb{R}$ and $f(x) = 1/x^2$ if $x \neq 0$, $f(0) = \infty$. Let $A \subset \mathbb{R}$ and $x \in X$. Find $\mu_f(A)$ in the following cases: $A = \{x\}$; $A = \mathbb{N}$; $A = \mathbb{Z}$; $A = (0, 1]$.

Comment. It might not be obvious to everybody what we actually mean by the sum $\sum_{x \in A} f(x)$. The definition goes as follows :

If A is empty, we sum over nothing; so we set $\sum_{x \in A} f(x) := 0$ *in this case.*

Assume from now on that A is non-empty.

If $f(x) = \infty$ *for (at least one)* $x \in A$*, then the only reasonable value for the sum is* ∞*, so we set* $\sum_{x \in A} f(x) := \infty$ *in this case.*

Assume also from now on that $f(x) < \infty$ *for all* $x \in A$*.*

If A is finite, so we may enumerate $A = \{a_1, a_2, \ldots, a_n\}$ *(without repetitions),* $\sum_{x \in A} f(x)$ *has it obvious meaning, namely*

$$\sum_{x \in A} f(x) := \sum_{i=1}^{n} f(a_i)\,.$$

Finally, If A is infinite, then we set

$$\sum_{x \in A} f(x) := \sup\{ \sum_{x \in B} f(x) \,|\, B \subset A\,,\ B \text{ finite}\}.$$

This means that $\sum_{x \in A} f(x) < \infty$ *whenever the set of real numbers* $\{ \sum_{x \in B} f(x) \,|\, B \subset A\,,\ B \text{ finite}\} \subset [0, \infty)$ *has an upper bound (in which case* $\sum_{x \in A} f(x)$ *is the least upper bound of this set), while* $\sum_{x \in A} f(x) = \infty$ *otherwise.*

For example, in the special case when A is countably infinite, so we may enumerate $A = \{a_1, a_2, \ldots\}$ *(without repetitions), one almost immediate sees that*

$$\sum_{x \in A} f(x) = \sum_{i=1}^{\infty} f(a_i)\,.$$

Remarks. Note that μ_f gives the counting measure on $\mathcal{P}(X)$ if one chooses $f = 1$, that is, $f(x) = 1$ for all $x \in X$.

If $x_0 \in X$ and we set $f = 1_{\{x_0\}}$, we then get a measure which is usually denoted by δ_{x_0} and called the *Dirac measure* on $\mathcal{P}(X)$. It satisfies:

$\delta_{x_0}(A) = 1$ if $x_0 \in A$, while $\delta_{x_0}(A) = 0$ if $x_0 \notin A$.

Note also that if $\mathcal{A}$ is a σ-algebra in X, then we may always restrict μ_f to $\mathcal{A}$ and thereby obtain a measure on $\mathcal{A}$. This means that there are always exist many such measures on $\mathcal{A}$. However, we will see later on that there are usually many other measures on $\mathcal{A}$ which are not given by this procedure.