MAT4300 - Fall 09 - Old exams, set 1.

Notation : we let $\mathcal{B}$ denote the σ-algebra of Borel subsets of $\mathbb{R}$ and λ be the Lebesgue measure on $\mathcal{B}$. If $E \in \mathcal{B}$, $\mathcal{B}_E$ consists of all Borel subsets of E and λ_E is the restriction of λ to $\mathcal{B}_E$.

Exercise 1, MAT4300, Fall 2008

Consider the measure μ on $\mathcal{A} = \mathcal{B}_{[0,2]}$ given by

$$\mu(A) = \int_A x^3 \, d\lambda_{[0,2]}(x) \, , \; A \in \mathcal{A} \, .$$

Find $\int x^4 \, d\mu(x)$.

Exercise 3, MAT4300, Fall 2008.

Set $\mathcal{A} = \mathcal{B}_{[0,1]}$ and let $\mu = \lambda_{[0,1]} \times \lambda_{[0,1]}$ be the product measure on $\mathcal{A} \otimes \mathcal{A}$.

a) Let L be a line-segment in $[0,1] \times [0,1]$. Explain why $L \in \mathcal{A} \otimes \mathcal{A}$ and $\mu(L) = 0$.

b) Let $F : [0,1] \times [0,1] \in \mathbb{R}$ be given by

$$F(x,y) = |\, x^2 - y \,| \, .$$

Show that $F \in \mathcal{L}^1(\mu)$ and compute $\int F \, d\mu$.

c) For each $n \in \mathbb{N}$, $x, y \in [0,1]$, define $F_n(x,y)$ by

$F_n(x,y) = |\, (x + \frac{1}{n})^2 - (y - \frac{1}{n}) \,|$ if x is irrational, while

$F_n(x,y) = x^{\frac{1}{n}} \, y^n$ if x is rational.

Explain why each $F_n \in \mathcal{L}^1(\mu)$ and find $\lim_{n \to \infty} \int F_n \, d\mu$.

Exercise 1, MAT4300, Fall 2007.

a) Define $F : \mathbb{R} \to \mathbb{R}$ by $F(x) = x^{-\frac{2}{3}} \, \mathbf{1}_{[-1,1]}(x)$ and set $\mu = F \cdot \lambda$. Compute $\mu(\mathbb{R})$ and find

$$\lim_{n \to \infty} n \int \sin\left(\frac{x}{n}\right) d\mu(x) \, .$$

b) What does one mean by the Lebesgue decomposition of λ w. r. to μ ? Find it.

Exercise 3 a) b), MAT4300, Fall 2006.

a) For each $n \in \mathbb{N}$, define $f_n : (0,1) \to \mathbb{R}$ by

$$f_n(x) = \frac{\sin(x^n)}{x^n}.$$

Find $\lim_{n\to\infty} \int f_n \, d\lambda_{(0,1)}$.

(You might here use the fact that $t \to \frac{\sin(t)}{t}$ is decreasing on (0,1)).

For each $n \in \mathbb{N}$, define $g_n : [-\frac{1}{2}, 1] \to \mathbb{R}$ by

$$g_n(x) = \frac{\sin(x^n)}{x^{n-1}}\,,\ x \neq 0\,,$$

and $g_n(0) = 1$.

Find $\lim_{n\to\infty} \int g_n \, d\lambda_{[-\frac{1}{2},1]}$.

b) For each $n \in \mathbb{N}$, define $f_n : [1,\infty) \to \mathbb{R}$ by

$$f_n(x) = \frac{n}{n\,x^{\frac{1}{3}} + 1}\,.$$

Show that $f_n \in \mathcal{L}^p(\lambda_{[1,\infty)})$ whenever $3 < p < \infty$.

Assume that $3 < p < \infty$. Decide whether the sequence $\{f_n\}_{n\in\mathbb{N}}$ is convergent in $\mathcal{L}^p(\lambda_{[1,\infty)})$ and find its limit if it converges.