**MAT4300 - Fall 09 - Old exams, set 2.**

Notation : we recall that if $\mathcal{B} = \mathcal{B}(\mathbb{R})$ denotes the $\sigma$-algebra of Borel subsets of $\mathbb{R}$, $\lambda$ is the Lebesgue measure on $\mathcal{B}$ and $E \in \mathcal{B}$, then $\mathcal{B}_E$ denotes all Borel subsets of $E$ and $\lambda_E$ denotes the restriction of $\lambda$ to $\mathcal{B}_E$.

**Exercise 2, MAT4300, Fall 2006**

Let $\mu$ be the counting measure on the $\sigma$-algebra $\mathcal{A} = \mathcal{P}(\mathbb{N})$.
Set $\mathbb{J} = \{2n \,|\, n \in \mathbb{N}\}$ and define a measure $\nu$ on $\mathcal{A}$ by

$$\nu(A) = \sum_{n \in A \cap \mathbb{J}} \frac{1}{n^2}\,.$$

a) Is $\mu << \nu$ ? Is $\nu << \mu$? Describe the Radon-Nykodym derivative when the answer is positive.

b) Find the Lebesgue decomposition of $\mu$ w. r. to $\nu$.

**Exercise 4, MAT4300, Fall 2006 and Exercise 4, Fall 2004** (both slightly reformulated).

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ denote $\sigma$-finite measure spaces and $\mu \times \nu$ denote the product measure on $\mathcal{C} = \mathcal{A} \otimes \mathcal{B}$.

Let $f \in \mathcal{M}^+(\mathcal{A}), g \in \mathcal{M}^+(\mathcal{B})$.

a) Consider the $\sigma$-finite measures $\mu_f = f \cdot \mu$ on $\mathcal{A}$ and $\nu_g = g \cdot \nu$ on $\mathcal{B}$, and the associated product measure $\mu_f \times \nu_g$ on $\mathcal{C}$ .

Define $F : X \times Y \to \mathbb{R}$ by $F(x, y) = f(x)\, g(y)$.

Show that $F \in \mathcal{M}^+(\mathcal{C})$ and $\mu_f \times \nu_g = F \cdot (\mu \times \nu)$.

b) Assume that $(Y, \mathcal{B}, \nu) = ([0, \infty), \mathcal{B}_{[0,\infty)}, \lambda_{[0,\infty)})$. Show that

$$\int f^2 \, d\mu = \int 2y \, \mu(\{f \geq y\}) \, d\lambda_{[0,\infty)}(y)\,.$$

*Hint* : Consider the function $H \in \mathcal{M}^+(\mathcal{C})$ given by $H(x, y) = 2y \, \mathbf{1}_{[0,f(x)]}(y)$ $( = 2y\, \mathbf{1}_E(x, y)$ where $E = \{(x, y) \in X \times [0, \infty) \,|\, 0 \leq y \leq f(x)\} \in \mathcal{C})$.
Then integrate $H$ w. r. to $\mu \times \lambda_{[0,\infty)}$.

c) Assume now that $(Y, \mathcal{B}, \nu) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$.

Define $G : X \times \mathbb{R} \to \mathbb{R}$ by $G(x, y) = f(x)y - y^2$.

Let $A \in \mathcal{A}$ and set $B = \{(x, y) \in X \times \mathbb{R} \mid 0 \leq y \leq f(x)\,,\ x \in A\} \in \mathcal{C}$.

Note that $G \geq 0$ on $B$. Show that

$$\int_B G\, d(\mu \times \lambda) = \frac{1}{6} \int_A f(x)^3\, d\mu(x)\,.$$

**Exercise 2, MAT4300, Fall 2005** (slightly reformulated)

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \mathbf{1}_{(n,n+1)}$ .

a) Is $f \in \mathcal{L}^1(\lambda)$ ? Does the improper Riemann-integral $\int_0^{\infty} f(x)dx$ converges ?

b) Set $g(x, y) = x\, f(x)\, e^{-xy}$ , $(x, y) \in \mathbb{R}^2$. Find $g^+$ and $g^-$.

c) Is $g \in \mathcal{L}^1(\lambda \times \lambda)$ ? Set $E = [1, \infty) \times [1, \infty)$. Is $1_E\, g \in \mathcal{L}^1(\lambda \times \lambda)$?