MAT4300: Mandatory Assignment, Fall 2010

Deadline: *You must turn in your paper before 2.30 p.m Thursday, November 4, on the 7th floor NHA. Remember to use the official front page available at*

http://www.math.uio.no/academics/obligforsideMI.pdf

If you due to illness or other circumstances want to extend the deadline, you must apply for an extension to studieinfo@math.uio.no. *Remember that illness has to be documented by a medical doctor! See*

http://www.uio.no/studier/emner/matnat/math/MAT4300/h10/obliger.xml

for more information about the rules for mandatory assignments

Instructions: *The assignment is compulsory, and students who do not get their paper accepted, will not get access to the final exam. To get the assignment accepted, you need a score of at least 60%. In the evaluation, credit will be given for a clear and well organized presentation. All questions (points a), b) etc.) have equal weight. Students who do not get their original paper accepted, but who have made serious and documented attempts to solve the problems, will get* one *chance of turning in an improved version.*

In solving the problems you may collaborate with others and use tools of all kinds. However, the paper you turn in should be written by you *(by hand or computer) and should reflect* your *understanding of the material. If we are not convinced that you understand your own paper, we may ask you to give an oral presentation.*

Let X be a non-empty set and $\mathcal{A}$ a collection of subsets of X. We call $\mathcal{A}$ an *algebra* if the following three conditions are satisfied:

(i) $\emptyset \in \mathcal{A}$

(ii) If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$

(iii) If $A, B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$

In this assignment, we always assume that $\mathcal{A}$ is an algebra on X.

a) Show that if $A_1, A_2, \ldots, A_n \in \mathcal{A}$, then $A_1 \cup A_2 \cup \ldots \cup A_n \in \mathcal{A}$

b) Show that if $A, B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$ and $A \setminus B \in \mathcal{A}$.

c) Let $\mathbb{N}$ be the set of natural numbers and

$$\mathcal{B} = \{B \subset \mathbb{N} \mid B \text{ or } B^c \text{ is finite}\}$$

Show that $\mathcal{B}$ is an algebra.

d) Describe the σ-algebra $\sigma(\mathcal{B})$ generated by $\mathcal{B}$.

A function $\mu : \mathcal{A} \to [0, \infty]$ is called a f*initely additive measure* if the following conditions are satisfied:

(i) $\mu(\emptyset) = 0$

(ii) $\mu(A \cup B) = \mu(A) + \mu(B)$ for all disjoint $A, B \in \mathcal{A}$

In the rest of the assignment, we assume that μ is a finitely additive measure on $\mathcal{B}$.

e) Show that if $A_1, A_2, \ldots, A_n \in \mathcal{A}$ are disjoint, then

$$\mu(A_1 \cup A_2 \cup \ldots \cup A_n) = \mu(A_1) + \mu(A_2) + \cdots + \mu(A_n)$$

f) Let $\mathbb{N}$ and $\mathcal{B}$ be as in point c) above. Define $\nu : \mathcal{B} \to [0.\infty]$ by

$$\nu(B) = \begin{cases} 0 & \text{if } B \text{ is finite} \\ \\ 1 & \text{if } B^c \text{ is finite} \end{cases}$$

Show that ν is a finitely additive measure.

g) Can ν be extended to a measure on the σ-algebra generated by $\mathcal{B}$?

h) Find a function $f : \mathbb{N} \to \mathbb{R}$ such that $f^{-1}(I) \in \mathcal{B}$ for all open intervals $I = (-\infty, \alpha)$, $\alpha \in \mathbb{R}$, but not for all closed intervals $I = (-\infty, \alpha]$, $\alpha \in \mathbb{R}$.

Point h) indicates that it is problematic to come up with a good notion of measurable function in the finitely additive case, and integrals must therefore be defined differently than in the countably additive case. To make the arguments simpler, we shall from now on assume that μ is bounded, i.e. $\mu(X) < \infty$.

A *simple function on standard form* is simply a function $f : X \to \mathbb{R}$ on the form $f = \sum_{i=1}^{n} a_i \mathbf{1}_{A_i}$, where A_i are disjoint set from $\mathcal{A}$. Just as in the countably infinite case, one may show that if $f = \sum_{j=1}^{m} b_j \mathbf{1}_{B_j}$ is another standard form representation of f, then $\sum_{i=1}^{n} a_i \mu(A_i) = \sum_{j=1}^{m} b_j \mu(B_j)$ (you need not show this). For a simple function $f = \sum_{i=1}^{n} a_i \mathbf{1}_{A_i}$, we define the *preliminary integral* $I(f)$ by $I(f) = \sum_{i=1}^{n} a_i \mu(A_i)$.

i) Assume that f, g are step functions such that $f \leq g$. Show that $I(f) \leq I(g)$.

For nonnegative, bounded functions $h : X \to \mathbb{R}$, define the *upper* and *lower integral* of h by

$$\overline{\int} h \, d\mu = \inf\{I(f) \mid f \text{ is a simple function } h \leq f\}$$

$$\underline{\int} h\, d\mu = \sup\{I(f) \mid f \text{ is a simple function } 0 \leq f \leq h\}$$

We say that h is *integrable* if $\overline{\int} h\, d\mu = \underline{\int} h\, d\mu$, and we then define the integral of h by

$$\int h\, d\mu = \overline{\int} h\, d\mu = \underline{\int} h\, d\mu$$

j) Show that if f is a nonnegative, simple function, then f is integrable and $\int f\, d\mu = I(f)$.

k) Let $h : \mathbb{N} \to \mathbb{R}$ be the function $h(n) = \frac{1}{n}$. Show that h is integrable and find $\int h\, d\nu$.

l) Let $(Y, \mathcal{C}, \lambda)$ be a measure space (in the ordinary sense). Show that if $\lambda(Y) \neq 0$, and $f : Y \to \mathbb{R}$ is a strictly positive, measurable function, then $\int f\, d\lambda > 0$.