Solution to: MAT4300 Mandatory Assignment, Fall 2010

a) We use induction on the number of sets. The condition is obviously satisfied for $n = 1$ and $n = 2$, and if it holds for $n = k$, the following argument shows that it holds for $n = k + 1$:

$$A_1 \cup A_2 \cup \ldots \cup A_{k+1} = (A_1 \cup A_2 \cup \ldots \cup A_k) \cup A_{k+1} \in \mathcal{A}$$

since $A_1 \cup A_2 \cup \ldots \cup A_k$ is in $\mathcal{A}$ by the induction hypothesis, and the union of two sets in $\mathcal{A}$ is in $\mathcal{A}$ by property (iii).

b) By De Morgan's laws, $A \cap B = (A^c \cup B^c)^c$ which is in $\mathcal{A}$ since $\mathcal{A}$ is closd under complements and finite unions. Since $A \setminus B = A \cap B^c$, it follows that $A \setminus B \in \mathcal{A}$

c) Call a set *cofinite* if it has finite complement. We check the three conditions:

(i) $\emptyset \in \mathcal{B}$ since it is finite.

b) If A is finite, A^c is cofinite, and if A is cofinite, A^c is finite. In either case, $A^c \in \mathcal{B}$.

c) If both A and B are finite, then $A \cup B$ is finite, and hence in $\mathcal{A}$. If one of the sets A, B is cofinite, so is the (even bigger) set $A \cup B$, and hence $A \cup B \in \mathcal{B}$.

d) Any subset C of $\mathbb{N}$ is countable, and hence the countable union of singletons (sets with only one element). Since the singletons are in $\mathcal{B}$, $C \in \sigma(\mathcal{B})$. This means that $\sigma(\mathcal{B}) = \mathcal{P}(\mathbb{N})$, the set of all subsets of $\mathbb{N}$.

e) We prove this by induction on the number of sets. The condition obviously holds for $n = 1$ and $n = 2$, and if it holds for $n = k$, the following argument shows that it holds for $n = k + 1$:

$$\mu(A_1 \cup A_2 \cup \ldots \cup A_{k+1}) = \mu((A_1 \cup A_2 \cup \ldots \cup A_k) \cup A_{k+1}) =$$

$$= \mu(A_1 \cup A_2 \cup \ldots \cup A_k) + \mu(A_{k+1}) = \mu(A_1) + \mu(A_2) + \cdots + \mu(A_k) + \mu(A_{k+1})$$

where we first used property (ii) and then the induction hypothesis.

f) Since $\emptyset$ is finite, $\nu(\emptyset) = 0$, and hence it suffices to prove that $\nu(A \cup B) = \nu(A) + \nu(B)$ for all disjoint $A, B \in \mathcal{B}$. Since two cofinite sets can not be disjoint, it suffices to look at the cases where one or both of the sets A and B are finite. If both the sets are finite, so is $A \cup B$, and hence $\nu(A \cup B) = 0$ and $\nu(A) + \nu(B) = 0 + 0 = 0$. If one of the sets is finite and the other is cofinite, then $A \cup B$ is cofinite, and hence $\nu(A \cup B) = 1$ and $\nu(A) + \nu(B) = 1 + 0 = 1$.

g) No. Assume μ is such an extension, then

$$1 = \mu(\mathbb{N}) = \mu\left(\bigcup_{n\in\mathbb{N}}\{n\}\right) = \sum_{n\in\mathbb{N}}\mu(\{n\}) = \sum_{n\in\mathbb{N}} 0 = 0$$

Comment: Some have attempted to solve this problem by showing that the conditions of Caratheodory's Extension Theorem is not satisfied. In general, this is not a good strategy — the conclusion of a theorem may hold even if the conditions fail!

h) Define

$$f(n) = \begin{cases} \frac{1}{n} & \text{if } n \text{ is odd} \\ \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Then $f^{-1}((-\infty,\alpha))$ is empty if $\alpha \leq 0$ and cofinite if $\alpha > 0$, and hence in $\mathcal{B}$ in both cases. However, $f^{-1}((-\infty,0])$ is the set of even numbers which is not in $\mathcal{B}$.

i) Let $f = \sum_{i=1}^{n} a_i \mathbf{1}_{A_i}$ and $g = \sum_{j=1}^{m} b_j \mathbf{1}_{B_j}$ where we assume that $X = \bigcup_{i=1}^{n} A_i = \bigcup_{j=1}^{m} B_j$. Then $f = \sum_{i,j} a_i \mathbf{1}_{A_i\cap B_j}$ and $g = \sum_{i,j} b_j \mathbf{1}_{A_i\cap B_j}$ are also simple function representations of f and g. If $A_i \cap B_j \neq \emptyset$, we must have $a_i \leq b_j$ since $f \leq g$, and hence

$$I(f) = \sum_{i,j} a_i \mu(A_i \cap B_j) \leq \sum_{i,j} b_j \mu(A_i \cap B_j) = I(g)$$

j) From i) we see that

$$\overline{\int} f\, d\mu = \inf\{I(g) \mid g \text{ is a simple function } f \leq g\} = I(f)$$

$$\underline{\int} f\, d\mu = \sup\{I(g) \mid g \text{ is a simple function } 0 \leq g \leq f\} = I(f)$$

since f is a simple function in the sets we are taking inf and sup over. Hence f is integrable, and $\int f\, d\mu = I(f)$.

k) For all $N \in \mathbb{N}$, the function $f = \mathbf{1}_{\{1,2,\ldots,n-1\}} + \frac{1}{n}\mathbf{1}_{\{n,n+1\ldots\}}$ is a step function majorizing h, and

$$I(f) = 1 \cdot \nu(\{1,2,\ldots,n-1\}) + \frac{1}{n}\nu(\{n.n+1,\ldots\}) = 0 + \frac{1}{n} = \frac{1}{n}$$

This means that $\overline{\int} h\, d\nu \leq 0$. On the other hand, the constant function $g \equiv 0$ is a lower approximation to h, and since $I(g) = 0$, $\underline{\int} h\, d\nu \geq 0$. Since we always have $\underline{\int} h\, d\mu \leq \overline{\int} h\, d\nu$, this means that h is integrable and $\int h\, d\nu = 0$.

Comment: Some try to use Beppo Levi's Theorem here, but we have only proved the theorem for countably additive measures, and here we are in a finitely additive situation.

l) Obviously, $Y = \bigcup_{n\in\mathbb{N}} f^{-1}((\frac{1}{n}, \infty))$, and by continuity of measures,

$$\lim_{n\to\infty} \lambda\left(f^{-1}((\frac{1}{n}, \infty))\right) = \lambda(Y) > 0$$

Hence there must be an n such that $\alpha := \lambda\left(f^{-1}((\frac{1}{n}, \infty))\right) > 0$. But then $g = \frac{1}{n}\mathbf{1}_{\{f^{-1}((\frac{1}{n},\infty))\}}$ is a step function majorized by f, and thus

$$\int f\, d\lambda \geq \int g\, d\lambda) = \frac{1}{n}\lambda\left(f^{-1}((\frac{1}{n}, \infty))\right) = \frac{1}{n}\alpha > 0$$