Solutions to exam in MATH4300, fall 2010

Problem 1: The function $|f|$ is integrable, and $|\mathbf{1}_{[-n,n]}f|$ is bounded by $|f|$ for all $n \in \mathbb{N}$. Hence by the Lebesgue's Dominated Convergence Theorem

$$\lim_{n\to\infty} \int_{[-n,n]} f \, d\lambda = \lim_{n\to\infty} \int \mathbf{1}_{[-n,n]} f \, d\lambda = \int \lim_{n\to\infty} \mathbf{1}_{[-n,n]} f \, d\lambda = \int f \, d\lambda$$

Problem 2: a) For $n = 1$ there is nothing to prove, and for $n = 2$ this is just property (ii) of the definition. Assume the property holds for $n = k$, we shall show that it holds for $n = k + 1$:

$$I(f_1 + f_2 + \cdots + f_{k+1}) = I((f_1 + f_2 + \cdots + f_k) + f_{k+1}) =$$

$$= I(f_1 + f_2 + \cdots + f_k) + I(f_{k+1}) = I(f_1) + I(f_2) + \cdots + I(f_k) + I(f_{k+1})$$

where we first used property (ii) and then the induction hypothesis.

b) Since $g - f \in \mathcal{M}^+$, we have

$$I(g) = I((g - f) + f) = I(g - f) + I(f) \geq I(f)$$

since $I(g - f) \geq 0$.

c) We have to check that $\mu(\emptyset) = 0$ and that $\mu(\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n\in\mathbb{N}} \mu(E_n)$ for all disjoint sequences $\{E_n\}$ of sets from $\mathcal{A}$.

For the first part, observe that

$$\mu(\emptyset) = I(\mathbf{1}_\emptyset) = I(0 \cdot \mathbf{1}_\emptyset) = 0 \cdot I(\mathbf{1}_\emptyset) = 0$$

where we have used property (i). For the second part, we note that

$$\mu(\bigcup_{n\in\mathbb{N}} E_n) = I(\mathbf{1}_{\bigcup_{n\in\mathbb{N}} E_n}) = I(\lim_{N\to\infty} \mathbf{1}_{\bigcup_{n=1}^N E_n}) = \lim_{N\to\infty} I(\mathbf{1}_{\bigcup_{n=1}^N E_n}) =$$

$$= \lim_{N\to\infty} I(\sum_{n=1}^N \mathbf{1}_{E_n}) = \lim_{N\to\infty} \sum_{n=1}^N I(\mathbf{1}_{E_n}) = \sum_{n=1}^\infty I(\mathbf{1}_{E_n}) = \sum_{n=1}^\infty \mu(E_n)$$

where we have used property (iii) to pull the limit outside I and part a) to get I inside the finite sum.

d) If $f = \sum_{i=1}^n \alpha_i \mathbf{1}_{E_i}$ is a positive, simple function

$$I(f) = I(\sum_{i=1}^n \alpha_i \mathbf{1}_{E_i}) = \sum_{i=1}^n I(\alpha_i \mathbf{1}_{E_i}) = \sum_{i=1}^n \alpha_i I(\mathbf{1}_{E_i}) = \sum_{i=1}^n \alpha_i \mu(E_i) = \int f \, d\mu$$

where we have used a), (i), c) and the definition of the integral for simple functions.

e) Let $\{f_n\}$ be an increasing sequence of simple functions converging to f. By Beppo Levi's Theorem, $\int f\, d\mu = \lim_{n\to\infty} \int f_n\, d\mu$ and by property (iii), $I(f) = \lim_{n\to\infty} I(f_n)$. By the previous point, $\int f_n\, d\mu = I(f_n)$ and hence $\int f\, d\mu = I(f)$

Problem 3: a) The sets are equal if $\mathbf{a} = \mathbf{b}$. If one of the sequences is an extension of the other, then the set belonging to the the longer sequence is properly contained in the other. The only remaining possibility is that $a_i \neq b_i$ for some i, and in this case $C_{\mathbf{a}}$ and $C_{\mathbf{b}}$ are disjoint.

b) We have to check the three points in the definition of a semi-ring:

(i) $\emptyset \in \mathcal{C}$ by definition.

(ii) Assume $S, T \in \mathcal{C}$. According to a), the intersection $S \cap T$ is either empty or equal to either S or T. In both cases $S \cap T \in \mathcal{C}$.

(iii) Assume $S, T \in \mathcal{C}$. If either S or T is empty, there is nothing to prove. If S and T are cylinder sets, we have to check the three cases in a). First note that if S and T are equal, or S is contained in T, then $S \setminus T = \emptyset \in \mathcal{C}$. If S and T are disjoint, then $S \setminus T = S \in \mathcal{C}$. In the only remaining case, T is contained in S, which means thsat $S = C_{\mathbf{a}}$ and $T = C_{\mathbf{b}}$ where $\mathbf{b}$ is an extension of $\mathbf{a}$. But then

$$S \setminus T = \bigcup_{\mathbf{e} \in E} C_{\mathbf{e}}$$

where E is the set of all other extensions of $\mathbf{a}$ with the same length as $\mathbf{b}$.

c) Obviously, $C_{\mathbf{a}} = C_{\mathbf{a}0} \cup C_{\mathbf{a}1}$. Any other cylinder set contained in $C_{\mathbf{a}}$ must be properly contained in either $C_{\mathbf{a}0}$ or $C_{\mathbf{a}0}$, and cannot make up all of $C_{\mathbf{a}}$ with just one other cylinder set.

d) From c) we know that if $C = C_{\mathbf{a}}$, then D and E must be $C_{\mathbf{a}0}$ and $C_{\mathbf{a}1}$. If $\mathbf{a}$ has length n, then $\rho(C) = 2^{-n}$, $\rho(D) = 2^{-(n+1)}$, $\rho(E) = 2^{-(n+1)}$, and hence $\rho(C) = \rho(D) + \rho(E)$.

For the general case, we use the induction hypothesis:

$P(k)$: If a cylinder set C is the disjoint union of k or fewer cylinder sets $C_1, C_2, \ldots, C_i$, then $\rho(C) = \rho(C_1) + \rho(C_2) + \cdots + \rho(C_i)$

We have already seen that $P(2)$ holds. Assume that $P(k)$ holds, and that the cylinder set C is the union of $k + 1$ cylinder set:

$$C = C_1 \cup C_2 \cup \ldots \cup C_{k+1}$$

If $C = C_{\mathbf{a}}$, the sets $C_1, C_2, \ldots, C_{k+1}$ fall into two groups; those that are subsets of $C_{\mathbf{a}0}$, and those that are subsets of $C_{\mathbf{a}1}$. In each category, there are k or less sets, and by the induction hypothesis, $\rho(C_{\mathbf{a}0})$ and $\rho(C_{\mathbf{a}1})$ are the sum of ρ applied to their respective subsets. But then

$$\rho(C_{\mathbf{a}}) = \rho(C_{\mathbf{a}0}) + \rho(C_{\mathbf{a}1}) = \rho(C_1) + \rho(C_2) + \cdots + \rho(C_{k+1})$$

e) By Caratheodory's Extension Theorem we only need to check that $\rho(\emptyset) = 0$ and that whenever a disjoint, countable union $\bigcup_{n\in\mathbb{N}} C_n$ of sets in $\mathcal{C}$ happens to be in $\mathcal{C}$, then

$$\rho(\bigcup_{n\in\mathbb{N}} C_n) = \sum_{n\in\mathbb{N}} \rho(C_n)$$

The first condition is part of the definition of ρ, and the second follows from the claim and the previous point since a union $\bigcup_{n\in\mathbb{N}} C_n$ can only belong to $\mathcal{C}$ when it is actually finite.

f) Assume we have a potential winner $\mathbf{a}$ with extensions in $C \setminus \bigcup_{n=1}^{N} C_n$ for all N. Then either $\mathbf{a}0$ of $\mathbf{a}1$ (or both) must be a potential winner — if not, there would be numbers $N_0, N_1 \in \mathbb{N}$ such that $\mathbf{a}0$ had no extensions in $C \setminus \bigcup_{n=1}^{N_0} C_n$ and $\mathbf{a}1$ had no extensions in $C \setminus \bigcup_{n=1}^{N_1} C_n$, and then $\mathbf{a}$ would have no extensions in $C \setminus \bigcup_{n=1}^{N} C_n$ where $N = \max\{N_0, N_1\}$. Using this argument inductively, we get a sequence of potential winners, $\{\mathbf{a}_k\}$, each extending the previous. This sequence defines an element $\mathbf{a} = \{a_1, a_2, a_3, \ldots\} \in X$. For all N, $\mathbf{a} \in C \setminus \bigcup_{n=1}^{N} C_n$ (this is because each $\mathbf{a}_k$ has extensions in $C \setminus \bigcup_{n=1}^{N} C_n$, and since $C, C_1, \ldots, C_n$ are cylinder sets, all sequences that agree on sufficiently large initial segments are either both inside $C \setminus \bigcup_{n=1}^{N} C_n$ or both outside). Consequently, $\mathbf{a} \in C \setminus \bigcup_{n=1}^{\infty} C_n$ and we have our contradiction.

For those who know topology, there is an alternative way to prove the claim. Give the set $\{0, 1\}$ the discrete topology, and $X = \{0, 1\}^{\mathbb{N}}$ the product topology. By Tychonov's Theorem, X is compact. It is easy to check that cylinder sets are both open and closed (and hence compact). If a cylinder set is the countable union of other cylinder sets, we have an open covering of a compact set by open sets, and hence there is a finite subcover. This means that the union is actually finite, and the claim is proved.