Problen 4.6: Assume that $\{\mu_n\}_{n\in\mathbb{N}}$ is a sequence of measure on $(X,\mathcal{A})$, and that $\{\alpha_n\}_{n\in\mathbb{N}}$ is a sequence of positive real numbers. We shall show that

$$\nu(A)=\sum_{n=1}^{\infty}\alpha_n\mu_n(A)$$

defines a measure on $(X,\mathcal{A})$. We need to show that:

(i) $\nu(\emptyset)=0$

(ii) $\nu(\bigcup_{m=1}^{\infty}A_m)=\sum_{m=1}^{\infty}\nu(A_m)$ for all disjoint sequences $\{A_m\}_{m\in\mathbb{N}}$ from $\mathcal{A}$.

The first point is trivial

$$\nu(\emptyset)=\sum_{n=1}^{\infty}\alpha_n\mu_n(\emptyset)=\sum_{n=1}^{\infty}\alpha_n 0=0$$

For the second point, observe that

$$\nu(\bigcup_{m=1}^{\infty}A_m)=\sum_{n=1}^{\infty}\alpha_n\mu_n(\bigcup_{m=1}^{\infty}A_m)=\sum_{n=1}^{\infty}\alpha_n\sum_{m=1}^{\infty}\mu_n(A_m)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\alpha_n\mu_n(A_m)$$

On the other hand

$$\sum_{m=1}^{\infty}\nu(A_m)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha_n\mu_n(A_m)$$

and hence we have to prove that

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\alpha_n\mu_n(A_m)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha_n\mu_n(A_m)$$

There are several ways to show this, but I shall follow the suggestion in the book.

By definition

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\alpha_n\mu_n(A_m)=\lim_{i\to\infty}\sum_{n=1}^{i}\sum_{m=1}^{\infty}\alpha_n\mu_n(A_m)=$$

$$=\lim_{i\to\infty}\sum_{n=1}^{i}\lim_{j\to\infty}\sum_{m=1}^{j}\alpha_n\mu_n(A_m)=\lim_{i\to\infty}\lim_{j\to\infty}\sum_{n=1}^{i}\sum_{m=1}^{j}\alpha_n\mu_n(A_m)$$

If we put $\beta_{i,j}=\sum_{n=1}^{i}\sum_{m=1}^{j}\alpha_n\mu_n(A_m)$, we hence have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\alpha_n\mu_n(A_m)=\lim_{i\to\infty}\lim_{j\to\infty}\beta_{i,j}$$

By symmetry

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha_n\mu_n(A_m) = \lim_{j\to\infty}\lim_{i\to\infty}\beta_{i,j}$$

Since the sequence $\beta_{i,j}$ is increasing in both i and j, we may replace the limits by suprema

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\alpha_n\mu_n(A_m) = \sup_{i\in\mathbb{N}}\sup_{j\in\mathbb{N}}\beta_{i,j}$$

and

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha_n\mu_n(A_m) = \sup_{j\in\mathbb{N}}\sup_{i\in\mathbb{N}}\beta_{i,j}$$

It thus suffices to show that

$$\sup_{i\in\mathbb{N}}\sup_{j\in\mathbb{N}}\beta_{i,j} = \sup_{j\in\mathbb{N}}\sup_{i\in\mathbb{N}}\beta_{i,j}$$

This is a general property which holds for all index sets:

Proposition: *For any two sets I, J and all double sequences $\{\beta_{i,j}\}_{i\in I, j\in J}$ of real numbers*

$$\sup_{i\in I}\sup_{j\in J}\beta_{i,j} = \sup_{j\in J}\sup_{i\in I}\beta_{i,j}$$

Proof: We shall show that

$$\sup_{i\in I}\sup_{j\in J}\beta_{i,j} = \sup\{\beta_{i,j}\,|\,i\in I, j\in J\} \tag{1}$$

By symmetry, we then have

$$\sup_{j\in I}\sup_{i\in J}\beta_{i,j} = \sup\{\beta_{i,j}\,|\,i\in I, j\in J\}$$

and the equality in the proposition follows.

To prove (i), observe that for any fixed $i\in I$,

$$\sup_{j\in J}\beta_{i,j} \le \sup\{\beta_{i,j}\,|\,i\in I, j\in J\}$$

as we on the left are taking the supremum over a smaller set than on the right. But then

$$\sup_{i\in I}\sup_{j\in J}\beta_{i,j} \le \sup\{\beta_{i,j}\,|\,i\in I, j\in J\}$$

To prove the opposite inequality, let r be any number smaller than $\sup\{\beta_{i,j}\,|\,i\in I, j\in J\}$. There must be indicies $i_0\in I, j_0\in J$ such that $\beta_{i_0,j_0} > r$, and hence

$$\sup_{j\in J}\beta_{i_0,j} > r$$

But then

$$\sup_{i \in I} \sup_{j \in J} \beta_{i,j} > r$$

and since this holds for all $r < \sup\{\beta_{i,j} \mid i \in I, j \in J\}$, we must have

$$\sup_{i \in I} \sup_{j \in J} \beta_{i,j} \geq \sup\{\beta_{i,j} \mid i \in I, j \in J\}$$

As we already have the opposite inequality, (1) is proved.