1 Prologue.
Solutions to Problems 1.1–1.2

Problem 1.1 Name the figures on the left and right *Figure 1* and *Figure 2*, respectively. Figure 1 is a triangle but Figure 2 is a (convex) quadrangle: the 'hypotenuse' has a kink. This is easily seen by comparing in Figure 2 the slopes of the small triangle in the lower left (it is $2/5$) and the larger triangle on top (it is $3/8 \neq 2/5$).

Problem 1.2 We have to calculate the area of an isosceles triangle of side-length r, base b, height h and opening angle $\phi := 2\pi/2^j$. From elementary geometry we know that

$$\cos\tfrac{\phi}{2} = \tfrac{h}{r} \quad \text{and} \quad \sin\tfrac{\phi}{2} = \tfrac{b}{2r}$$

so that

$$\text{area (triangle)} = \frac{1}{2}\,hb = r^2 \cos\tfrac{\phi}{2}\sin\tfrac{\phi}{2} = \frac{r^2}{2}\,\sin\phi\,.$$

Since we have $\lim_{\phi\to 0}\frac{\sin\phi}{\phi} = 1$ we find

$$\begin{aligned}\text{area (circle)} &= \lim_{j\to\infty} 2^j\,\frac{r^2}{2}\,\sin\tfrac{2\pi}{2^j}\\ &= 2r^2\pi \lim_{j\to\infty}\frac{\sin\frac{2\pi}{2^j}}{\frac{2\pi}{2^j}}\\ &= 2r^2\pi\end{aligned}$$

just as we had expected.

2 The pleasures of counting. Solutions to Problems 2.1–2.21

Problem 2.1 (i) We have

$$\begin{aligned} x \in A \setminus B &\iff x \in A \text{ and } x \notin B \\ &\iff x \in A \text{ and } x \in B^c \\ &\iff x \in A \cap B^c. \end{aligned}$$

(ii) Using (i) and de Morgan's laws (*) yields

$$\begin{aligned} (A \setminus B) \setminus C &\overset{\text{(i)}}{=} (A \cap B^c) \cap C^c = A \cap B^c \cap C^c \\ &= A \cap (B^c \cap C^c) \overset{(*)}{=} A \cap (B \cup C)^c = A \setminus (B \cup C). \end{aligned}$$

(iii) Using (i), de Morgan's laws (*) and the fact that $(C^c)^c = C$ gives

$$\begin{aligned} A \setminus (B \setminus C) &\overset{\text{(i)}}{=} A \cap (B \cap C^c)^c \\ &\overset{(*)}{=} A \cap (B^c \cup C) \\ &= (A \cap B^c) \cup (A \cap C) \\ &\overset{\text{(i)}}{=} (A \setminus B) \cup (A \cap C). \end{aligned}$$

(iv) Using (i) and de Morgan's laws (*) gives

$$\begin{aligned} A \setminus (B \cap C) &\overset{\text{(i)}}{=} A \cap (B \cap C)^c \\ &\overset{(*)}{=} A \cap (B^c \cup C^c) \\ &= (A \cap B^c) \cup (A \cap C^c) \\ &\overset{\text{(i)}}{=} (A \setminus B) \cup (A \setminus C) \end{aligned}$$

(v) Using (i) and de Morgan's laws (*) gives

$$\begin{aligned} A \setminus (B \cup C) &\overset{\text{(i)}}{=} A \cap (B \cup C)^c \\ &\overset{(*)}{=} A \cap (B^c \cap C^c) \\ &= A \cap B^c \cap C^c \\ &= A \cap B^c \cap A \cap C^c \\ &\overset{\text{(i)}}{=} (A \setminus B) \cup (A \setminus C) \end{aligned}$$

Problem 2.2 Observe, first of all, that

$$A \setminus C \subset (A \setminus B) \cup (B \setminus C). \tag{$*$}$$

This follows easily from

$$\begin{aligned} A \setminus C &= (A \setminus C) \cap X \\ &= (A \cap C^c) \cap (B \cup B^c) \\ &= (A \cap C^c \cap B) \cup (A \cap C^c \cap B^c) \\ &\subset (B \cap C^c) \cup (A \cap B^c) \\ &= (B \setminus C) \cup (A \setminus B). \end{aligned}$$

Using this and the analogous formula for $C \setminus A$ then gives

$$\begin{aligned} &(A \cup B \cup C) \setminus (A \cap B \cap C) \\ &\quad= (A \cup B \cup C) \cap (A \cap B \cap C)^c \\ &\quad= [A \cap (A \cap B \cap C)^c] \cup [B \cap (A \cap B \cap C)^c] \cup [C \cap (A \cap B \cap C)^c] \\ &\quad= [A \setminus (A \cap B \cap C)] \cup [B \setminus (A \cap B \cap C)] \cup [C \setminus (A \cap B \cap C)] \\ &\quad= [A \setminus (B \cap C)] \cup [B \setminus (A \cap C)] \cup [C \setminus (A \cap B)] \\ &\quad\overset{(2.1(\text{iv}))}{=} (A \setminus B) \cup (A \setminus C) \cup (B \setminus A) \cup (B \setminus C) \cup (C \setminus A) \cup (C \setminus B) \\ &\quad\overset{(*)}{=} (A \setminus B) \cup (B \setminus A) \cup (B \setminus C) \cup (C \setminus B) \\ &\quad= (A \Delta B) \cup (B \Delta C) \end{aligned}$$

Problem 2.3 It is clearly enough to prove (2.3) as (2.2) follows if I contains 2 points. De Morgan's identities state that for any index set I (finite, countable or not countable) and any collection of subsets $A_i \subset X$, $i \in I$, we have

$$\text{(a)} \quad \Big(\bigcup_{i\in I} A_i\Big)^c = \bigcap_{i\in I} A_i^c \qquad \text{and} \qquad \text{(b)} \quad \Big(\bigcap_{i\in I} A_i\Big)^c = \bigcup_{i\in I} A_i^c.$$

In order to see (a) we note that

$$\begin{aligned} a \in \Big(\bigcup_{i\in I} A_i\Big)^c &\iff a \notin \bigcup_{i\in I} A_i \\ &\iff \forall\, i \in I \;:\; a \notin A_i \\ &\iff \forall\, i \in I \;:\; a \in A_i^c \\ &\iff a \in \bigcap_{i\in I} A_i^c, \end{aligned}$$

and (b) follows from

$$\begin{aligned} a \in \Big(\bigcap_{i\in I} A_i\Big)^c &\iff a \notin \bigcap_{i\in I} A_i \\ &\iff \exists\, i_0 \in I \;:\; a \notin A_{i_0} \\ &\iff \exists\, i_0 \in I \;:\; a \in A_{i_0}^c \\ &\iff a \in \bigcup_{i\in I} A_i^c. \end{aligned}$$

Problem 2.4 (i) The inclusion $f(A \cap B) \subset f(A) \cap f(B)$ is *always* true since $A \cap B \subset A$ and $A \cap B \subset B$ imply that $f(A \cap B) \subset f(A)$ and $f(A \cap B) \subset f(B)$, respectively. Thus, $f(A \cap B) \subset f(A) \cap f(B)$.

Furthermore, $y \in f(A) \setminus f(B)$ means that there is some $x \in A$ but $x \notin B$ such that $y = f(x)$, that is: $y \in f(A \setminus B)$. Thus, $f(A) \setminus f(B) \subset f(A \setminus B)$.

To see that the converse inclusions cannot hold we consider some *non injective* f. Take $X = [0, 2]$, $A = (0, 1)$, $B = (1, 2)$, and $f : [0, 2] \to \mathbb{R}$ with $x \mapsto f(x) = c$ (c is some constant). Then f is not injective and

$$\emptyset = f(\emptyset) = f((0, 1) \cap (1, 2)) \neq f((0, 1)) \cup f((1, 2)) = \{c\}.$$

Moreover, $f(X) = f(B) = \{c\} = f(X \setminus B)$ but $f(X) \setminus f(B) = \emptyset$.

(ii) Recall, first of all, the definition of f^{-1} for a map $f : X \to Y$ and $B \subset Y$

$$f^{-1}(B) := \{x \in X \;:\; f(x) \in B\}.$$

Observe that

$$\begin{aligned} x \in f^{-1}(\cup_{i\in I} C_i) &\iff f(x) \in \cup_{i\in I} C_i \\ &\iff \exists\, i_0 \in I \;:\; f(x) \in C_{i_0} \\ &\iff \exists\, i_0 \in I \;:\; x \in f^{-1}(C_{i_0}) \\ &\iff x \in \cup_{i\in I} f^{-1}(C_i), \end{aligned}$$

and

$$\begin{aligned} x \in f^{-1}(\cap_{i\in I} C_i) &\iff f(x) \in \cap_{i\in I} C_i \\ &\iff \forall\, i \in I \;:\; f(x) \in C_i \\ &\iff \forall\, i \in I \;:\; x \in f^{-1}(C_i) \\ &\iff x \in \cap_{i\in I} f^{-1}(C_i), \end{aligned}$$

and, finally,

$$\begin{aligned} x \in f^{-1}(C \setminus D) &\iff f(x) \in C \setminus D \\ &\iff f(x) \in C \quad \text{and} \quad f(x) \notin D \\ &\iff x \in f^{-1}(C) \quad \text{and} \quad x \notin f^{-1}(D) \\ &\iff x \in f^{-1}(C) \setminus f^{-1}(D). \end{aligned}$$

Problem 2.5

(i), (vi) For every x we have

$$\begin{aligned} \mathbf{1}_{A\cap B}(x) = 1 &\iff x \in A \cap B \\ &\iff x \in A,\ x \in B \\ &\iff \mathbf{1}_A(x) = 1 = \mathbf{1}_B(x) \\ &\iff \begin{cases} \mathbf{1}_A(x) \cdot \mathbf{1}_B(x) = 1 \\ \min\{\mathbf{1}_A(x), \mathbf{1}_B(x)\} = 1 \end{cases} \end{aligned}$$

(ii), (v) For every x we have

$$\begin{aligned} \mathbf{1}_{A\cup B}(x) = 1 &\iff x \in A \cup B \\ &\iff x \in A \text{ or } x \in B \\ &\iff \mathbf{1}_A(x) + \mathbf{1}_B(x) \geqslant 1 \\ &\iff \begin{cases} \min\{\mathbf{1}_A(x) + \mathbf{1}_B(x), 1\} = 1 \\ \max\{\mathbf{1}_A(x), \mathbf{1}_B(x)\} = 1 \end{cases} \end{aligned}$$

(iii) Since $A = (A \cap B) \uplus (A \setminus B)$ we see that $\mathbf{1}_{A\cap B}(x) + \mathbf{1}_{A\setminus B}(x)$ can never have the value 2, thus part (ii) implies

$$\begin{aligned} \mathbf{1}_A(x) = \mathbf{1}_{(A\cap B)\cup(A\setminus B)}(x) &= \min\{\mathbf{1}_{A\cap B}(x) + \mathbf{1}_{A\setminus B}(x), 1\} \\ &= \mathbf{1}_{A\cap B}(x) + \mathbf{1}_{A\setminus B}(x) \end{aligned}$$

and all we have to do is to subtract $\mathbf{1}_{A\cap B}(x)$ on both sides of the equation.

(iv) With the same argument that we used in (iii) and with the result of (iii) we get

$$\begin{aligned} \mathbf{1}_{A\cup B}(x) &= \mathbf{1}_{(A\setminus B)\cup(A\cap B)\cup(B\setminus A)}(x) \\ &= \mathbf{1}_{A\setminus B}(x) + \mathbf{1}_{A\cap B}(x) + \mathbf{1}_{B\setminus A}(x) \\ &= \mathbf{1}_A(x) - \mathbf{1}_{A\cap B}(x) + \mathbf{1}_{A\cap B}(x) + \mathbf{1}_B(x) - \mathbf{1}_{A\cap B}(x) \\ &= \mathbf{1}_A(x) + \mathbf{1}_B(x) - \mathbf{1}_{A\cap B}(x). \end{aligned}$$

Problem 2.6 (i) Using 2.5(iii), (iv) we see that

$$\begin{aligned}\mathbf{1}_{A\,\triangle\,B}(x) &= \mathbf{1}_{(A\setminus B)\cup(B\setminus A)}(x)\\ &= \mathbf{1}_{A\setminus B}(x) + \mathbf{1}_{B\setminus A}(x)\\ &= \mathbf{1}_{A}(x) - \mathbf{1}_{A\cap B}(x) + \mathbf{1}_{B}(x) - \mathbf{1}_{A\cap B}(x)\\ &= \mathbf{1}_{A}(x) + \mathbf{1}_{B}(x) - 2\mathbf{1}_{A\cap B}(x)\end{aligned}$$

and this expression is 1 if, and only if, x is either in A or B but not in both sets. Thus

$$\mathbf{1}_{A\,\triangle\,B}(x) \iff \mathbf{1}_{A}(x)+\mathbf{1}_{B}(x) = 1 \iff \mathbf{1}_{A}(x)+\mathbf{1}_{B}(x)\mathrm{mod}2 = 1.$$

It is also possible to show that

$$\mathbf{1}_{A\,\triangle\,B} = |\mathbf{1}_{A} - \mathbf{1}_{B}|.$$

This follows from

$$\mathbf{1}_{A}(x) - \mathbf{1}_{B}(x) = \begin{cases} 0, & \text{if } x \in A\cap B;\\ 0, & \text{if } x \in A^c\cap B^c;\\ +1, & \text{if } x \in A\setminus B;\\ -1, & \text{if } x \in B\setminus A.\end{cases}$$

Thus,

$$|\mathbf{1}_{A}(x) - \mathbf{1}_{B}(x)| = 1 \iff x \in (A\setminus B)\cup(B\setminus A) = A\,\triangle\,B.$$

(ii) From part (i) we see that

$$\begin{aligned}\mathbf{1}_{A\,\triangle\,(B\,\triangle\,C)} &= \mathbf{1}_{A} + \mathbf{1}_{B\,\triangle\,C} - 2\mathbf{1}_{A}\mathbf{1}_{B\,\triangle\,C}\\ &= \mathbf{1}_{A} + \mathbf{1}_{B} + \mathbf{1}_{C} - 2\mathbf{1}_{B}\mathbf{1}_{C} - 2\mathbf{1}_{A}\big(\mathbf{1}_{B} + \mathbf{1}_{C} - 2\mathbf{1}_{B}\mathbf{1}_{C}\big)\\ &= \mathbf{1}_{A} + \mathbf{1}_{B} + \mathbf{1}_{C} - 2\mathbf{1}_{B}\mathbf{1}_{C} - 2\mathbf{1}_{A}\mathbf{1}_{B} - 2\mathbf{1}_{A}\mathbf{1}_{C} + 4\mathbf{1}_{A}\mathbf{1}_{B}\mathbf{1}_{C}\end{aligned}$$

and this expression treats A, B, C in a completely symmetric way, i.e.

$$\mathbf{1}_{A\,\triangle\,(B\,\triangle\,C)} = \mathbf{1}_{(A\,\triangle\,B)\,\triangle\,C}.$$

(iii) **Step 1:** $(\mathcal{P}(X), \triangle, \emptyset)$ is an abelian group.
Neutral element: $A\,\triangle\,\emptyset = \emptyset\,\triangle\,A = A$;
Inverse element: $A\,\triangle\,A = (A\setminus A)\cup(A\setminus A) = \emptyset$, i.e. each element is its own inverse.
Associativity: see part (ii);

Commutativity: $A \triangle B = B \triangle A$.

Step 2: For the multiplication $\cap$ we have
Associativity: $A \cap (B \cap C) = (A \cap B) \cap C$;
Commutativity: $A \cap B = B \cap A$;
One-element: $A \cap X = X \cap A = A$.

Step 3: Distributive law:

$$A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C).$$

For this we use again indicator functions and the rules from (i) and Problem 2.5:

$$\begin{aligned}
\mathbf{1}_{A\cap(B \triangle C)} = \mathbf{1}_A \mathbf{1}_{B \triangle C} &= \mathbf{1}_A(\mathbf{1}_B + \mathbf{1}_C \mod 2) \\
&= \big[\mathbf{1}_A(\mathbf{1}_B + \mathbf{1}_C)\big] \mod 2 \\
&= \big[\mathbf{1}_A\mathbf{1}_B + \mathbf{1}_A\mathbf{1}_C\big] \mod 2 \\
&= \big[\mathbf{1}_{A\cap B} + \mathbf{1}_{A\cap C}\big] \mod 2 \\
&= \mathbf{1}_{(A\cap B) \triangle (A\cap C)}.
\end{aligned}$$

Problem 2.7 Let $f : X \to Y$. One has

$$\begin{aligned}
f \text{ surjective} &\iff \forall\, B \subset Y \ :\ f \circ f^{-1}(B) = B \\
&\iff \forall\, B \subset Y \ :\ f \circ f^{-1}(B) \supset B.
\end{aligned}$$

This can be seen as follows: by definition $f^{-1}(B) = \{x \,:\, f(x) \in B\}$ so that

$$f \circ f^{-1}(B) = f\big(\{x \,:\, f(x) \in B\}\big) = \{f(x) \,:\, f(x) \in B\} \subset \{y \,:\, y \in B\}$$

and we have equality in the last step if, and only if, we can guarantee that every $y \in B$ is of the form $y = f(x)$ for some x. Since this must hold for all sets B, this amounts to saying that $f(X) = Y$, i.e. that f is surjective. The second equivalence is clear since our argument shows that the inclusion '$\subset$' always holds.

Thus, we can construct a counterexample by setting $f : \mathbb{R} \to \mathbb{R}$, $f(x) := x^2$ and $B = [-1, 1]$. Then

$$f^{-1}([-1,1]) = [0,1] \quad\text{and}\quad f \circ f^{-1}([-1,1]) = f([0,1]) = [0,1] \subsetneq [-1,1].$$

On the other hand

$$\begin{aligned} f \text{ injective} &\iff \forall\, A \subset X \;:\; f^{-1} \circ f(A) = A \\ &\iff \forall\, A \subset X \;:\; f^{-1} \circ f(A) \subset A. \end{aligned}$$

To see this we observe that because of the definition of f^{-1}

$$f^{-1} \circ f(A) = \{x \;:\; f(x) \in f(A)\} \supset \{x \;:\; x \in A\} = A \qquad (*)$$

since $x \in A$ always entails $f(x) \in f(A)$. The reverse is, for non-injective f, wrong since then there might be some $x_0 \notin A$ but with $f(x_0) = f(x) \in f(A)$ i.e. $x_0 \in f^{-1} \circ f(A) \setminus A$. This means that we have equality in $(*)$ if, and only if, f is injective. The second equivalence is clear since our argument shows that the inclusion '$\supset$' always holds.

Thus, we can construct a counterexample by setting $f : \mathbb{R} \to \mathbb{R}$, $f \equiv 1$. Then

$$f([0,1]) = \{1\} \;\text{ and }\; f^{-1} \circ f([0,1]) = f^{-1}(\{1\}) = \mathbb{R} \supsetneq [0,1].$$

Problem 2.8 Assume that for x, y we have $f \circ g(x) = f \circ g(y)$. Since f is injective, we conclude that

$$f(g(x)) = f(g(y)) \implies g(x) = g(y),$$

and, since g is also injective,

$$g(x) = g(y) \implies x = y$$

showing that $f \circ g$ is injective.

Problem 2.9

- Call the set of odd numbers $\mathcal{O}$. Every odd number is of the form $2k - 1$ where $k \in \mathbb{N}$. We are done, if we can show that the map $f : \mathbb{N} \to \mathcal{O}$, $k \mapsto 2k-1$ is bijective. Surjectivity is clear as $f(\mathbb{N}) = \mathcal{O}$. For injectivity we take $i, j \in \mathbb{N}$ such that $f(i) = f(j)$. The latter means that $2i - 1 = 2j - 1$, so $i = j$, i.e. injectivity.
- The quickest solution is to observe that $\mathbb{N} \times \mathbb{Z} = \mathbb{N} \times \mathbb{N} \cup \mathbb{N} \times \{0\} \cup \mathbb{N} \times (-\mathbb{N})$ where $-\mathbb{N} := \{-n \;:\; n \in \mathbb{N}\}$ are the strictly negative integers. We know from Example 2.5(iv) that $\mathbb{N} \times \mathbb{N}$ is countable. Moreover, the map $\beta : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times (-\mathbb{N})$, $\beta((i,k)) = (i, -k)$ is bijective, thus $\#\mathbb{N} \times (-\mathbb{N}) = \#\mathbb{N} \times \mathbb{N}$ is also countable and so is $\mathbb{N} \times \{0\}$ since $\gamma : \mathbb{N} \to \mathbb{N} \times \{0\}$, $\gamma(n) := (n, 0)$ is also bijective.

  Therefore, $\mathbb{N} \times \mathbb{Z}$ is a union of three countable sets, hence countable.

An *alternative approach* would be to write out $\mathbb{Z}\times\mathbb{N}$ (the swap of $\mathbb{Z}$ and $\mathbb{N}$ is for notational reasons—since the map $\beta((j,k)) := (k,j)$ from $\mathbb{Z}\times\mathbb{N}$ to $\mathbb{N}\times\mathbb{Z}$ is bijective, the cardinality does not change) in the following form

$$\begin{matrix}
\dots & (-3,1) & (-2,1) & (-1,1) & (0,1) & (1,1) & (2,1) & (3,1) & \dots \\
\dots & (-3,2) & (-2,2) & (-1,2) & (0,2) & (1,2) & (2,2) & (3,2) & \dots \\
\dots & (-3,3) & (-2,3) & (-1,3) & (0,3) & (1,3) & (2,3) & (3,3) & \dots \\
\dots & (-3,4) & (-2,4) & (-1,4) & (0,4) & (1,4) & (2,4) & (3,4) & \dots \\
\dots & (-3,5) & (-2,5) & (-1,5) & (0,5) & (1,5) & (2,5) & (3,5) & \dots \\
\dots & (-3,6) & (-2,6) & (-1,6) & (0,6) & (1,6) & (2,6) & (3,6) & \dots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{matrix}$$

and going through the array, starting with $(0,1)$, then $(1,1) \to (1,2) \to (0,2) \to (-1,2) \to (-1,1)$, then $(2,1) \to (2,2) \to (2,3) \to (1,3) \to \dots$ in clockwise oriented $\sqcup$-shapes down, left, up.

- In Example 2.5(iv) we have shown that $\#\mathbb{Q} \leqslant \#\mathbb{N}$. Since $\mathbb{N}\subset\mathbb{Q}$, we have a canonical injection $\jmath : \mathbb{N}\to\mathbb{Q}, i\mapsto i$ so that $\#\mathbb{N}\leqslant\#\mathbb{Q}$. Using Theorem 2.7 we conclude that $\#\mathbb{Q} = \#\mathbb{N}$.

  The proof of $\#(\mathbb{N}\times\mathbb{N}) = \#\mathbb{N}$ can be easily adapted—using some pretty obvious notational changes—to show that the Cartesian product of any two countable sets of cardinality $\#\mathbb{N}$ has again cardinality $\#\mathbb{N}$. Applying this $m-1$ times we see that $\#\mathbb{Q}^n = \#\mathbb{N}$.

- $\bigcup_{m\in\mathbb{N}}\mathbb{Q}^m$ is a countable union of countable sets, hence countable, cf. Theorem 2.6.

Problem 2.10 Following the hint it is clear that $\beta : \mathbb{N}\to\mathbb{N}\times\{1\}, i\mapsto(i,1)$ is a bijection and that $\jmath : \mathbb{N}\times\{1\}\to\mathbb{N}\times\mathbb{N}, (i,1)\mapsto(i,1)$ is an injection. Thus, $\#\mathbb{N}\leqslant\#(\mathbb{N}\times\mathbb{N})$.

On the other hand, $\mathbb{N}\times\mathbb{N} = \bigcup_{j\in\mathbb{N}}\mathbb{N}\times\{j\}$ which is a countable union of countable sets, thus $\#(\mathbb{N}\times\mathbb{N})\leqslant\#\mathbb{N}$.

Applying Theorem 2.7 finally gives $\#(\mathbb{N}\times\mathbb{N}) = \#\mathbb{N}$.

Problem 2.11 Since $E\subset F$ the map $\jmath : E\to F, e\mapsto e$ is an injection, thus $\#E\leqslant\#F$.

Problem 2.12 Assume that the set $\{0,1\}^{\mathbb{N}}$ were indeed countable and that $\{s_j\}_{j\in\mathbb{N}}$ was an enumeration: each s_j would be a sequence of the form

$(d_1^j, d_2^j, d_3^j, ..., d_k^j, ...)$ with $d_k^j \in \{0,1\}$. We could write these sequences in an infinite list of the form:

$$\begin{array}{ccccccccc} s_1 & = & d_1^1 & d_2^1 & d_3^1 & d_4^1 & \dots & d_k^1 & \dots \\ s_2 & = & d_1^2 & d_2^2 & d_3^2 & d_4^2 & \dots & d_k^2 & \dots \\ s_3 & = & d_1^3 & d_2^3 & d_3^3 & d_4^3 & \dots & d_k^3 & \dots \\ s_4 & = & d_1^4 & d_2^4 & d_3^4 & d_4^4 & \dots & d_k^4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ s_k & = & d_1^k & d_2^k & d_3^k & d_4^k & \dots & d_k^k & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{array}$$

and produce a new 0-1-sequence $S = (e_1, e_2, e_3, \dots)$ by setting

$$e_m := \begin{cases} 0, & \text{if} \quad d_m^m = 1 \\ 1, & \text{if} \quad d_m^m = 0 \end{cases} .$$

Since S differs from s_ℓ exactly at position ℓ, S cannot be in the above list, thus, the above list did not contain all 0-1-sequences, hence a contradiction.

Problem 2.13 Consider the function $f : (0,1) \to \mathbb{R}$ given by

$$f(x) := \frac{1}{1-x} - \frac{1}{x}.$$

This function is obviously continuous and we have $\lim_{x\to 0} f(x) = -\infty$ and $\lim_{x\to 1} f(x) = +\infty$. By the intermediate value theorem we have therefore $f((0,1)) = \mathbb{R}$, i.e. surjectivity.

Since f is also differentiable and $f'(x) = \dfrac{1}{(1-x)^2} + \dfrac{1}{x^2} > 0$, we see that f is strictly increasing, hence injective, hence bijective.

Problem 2.14 Since $A_1 \subset \bigcup_{j\in\mathbb{N}} A_j$ it is clear that $\mathfrak{c} = \#A_1 \leqslant \#\bigcup_{j\in\mathbb{N}} A_j$. On the other hand, $\#A_j = \mathfrak{c}$ means that we can map A_j bijectively onto $\mathbb{R}$ and, using Problem 2.13, we map $\mathbb{R}$ bijectively onto $(0,1)$ or $(j-1,j)$. This shows that $\#\bigcup_{j\in\mathbb{N}} A_j \leqslant \#\bigcup_{j\in\mathbb{N}}(j-1,j) \leqslant \#\mathbb{R} = \mathfrak{c}$. Using Theorem 2.7 finishes the proof.

Problem 2.15 Since we can write each $x \in (0,1)$ as an infinite dyadic fraction (o.k. if it is finite, fill it up with an infinite tail of zeroes !), the proof of Theorem 2.8 shows that $\#(0,1) \leqslant \#\{0,1\}^{\mathbb{N}}$.

On the other hand, thinking in base-4 expansions, each element of $\{1,2\}^{\mathbb{N}}$ can be interpreted as a unique base-4 fraction (having no 0 or 3 in its expansion) of some number in $(0,1)$. Thus, $\#\{1,2\}^{\mathbb{N}} \leqslant \#\mathbb{N}$.

But $\#\{1,2\}^{\mathbb{N}} = \#\{0,1\}^{\mathbb{N}}$ and we conclude with Theorem 2.7 that $\#(0,1) = \#\{0,1\}^{\mathbb{N}}$.

Problem 2.16 Just as before, expand $x \in (0,1)$ as an n-adic fraction, then interpret each element of $\{1,2,\dots,n+1\}^{\mathbb{N}}$ as a unique $(n+1)$-adic expansion of a number in $(0,1)$ and observe that $\#\{1,2,\dots,n+1\}^{\mathbb{N}} = \{0,1,\dots,n\}^{\mathbb{N}}$.

Problem 2.17 Take a vector $(x,y) \in (0,1)\times(0,1)$ and expand its coordinate entries x, y as dyadic numbers:

$$x = 0.x_1x_2x_3\dots, \qquad y = 0.y_1y_2y_3\dots.$$

Then $z := 0.x_1y_1x_2y_2x_3y_3\dots$ is a number in $(0,1)$. Conversely, we can 'zip' each $z = 0.z_1z_2z_3z_4\dots \in (0,1)$ into two numbers $x, y \in (0,1)$ by setting

$$x := 0.z_2z_4z_6z_8\dots, \qquad y := 0.z_1z_3z_5z_7\dots$$

This is obviously a bijective operation.

Since we have a bijection between $(0,1) \leftrightarrow \mathbb{R}$ it is clear that we have also a bijection between $(0,1)\times(0,1) \leftrightarrow \mathbb{R}\times\mathbb{R}$.

Problem 2.18 We have seen in Problem 2.18 that $\#\{0,1\}^{\mathbb{N}} = \#\{1,2\}^{\mathbb{N}} = \mathfrak{c}$. Obviously, $\{1,2\}^{\mathbb{N}} \subset \mathbb{N}^{\mathbb{N}} \subset \mathbb{R}^{\mathbb{N}}$ and since we have a bijection between $(0,1) \leftrightarrow \mathbb{R}$ one extends this (using coordinates) to a bijection between $(0,1)^{\mathbb{N}} \leftrightarrow \mathbb{R}^{\mathbb{N}}$. Using Theorem 2.9 we get

$$\mathfrak{c} = \#\{1,2\}^{\mathbb{N}} \leqslant \#\mathbb{N}^{\mathbb{N}} \leqslant \#\mathbb{R}^{\mathbb{N}} = \mathfrak{c},$$

and, because of Theorem 2.7 we have equality in the above formula.

Problem 2.19 Let $F \in \mathcal{F}$ with $\#F = n$ Then we can write F as a tuple of length n (having n pairwise different entries...) and therefore we can interpret F as an element of $\bigcup_{m\in\mathbb{N}} \mathbb{N}^m$. In this sense, $\mathcal{F} \hookrightarrow \bigcup_{m\in\mathbb{N}} \mathbb{N}^m$ and $\#\mathcal{F} \leqslant \bigcup_{m\in\mathbb{N}} \mathbb{N}^m = \#\mathbb{N}$ since countably many countable sets are again countable. Since $\mathbb{N} \subset \mathcal{F}$ we get $\#\mathcal{F} = \#\mathbb{N}$ by Theorem 2.7.

Alternative: Define a map $\phi : \mathcal{F} \to \mathbb{N}$ by

$$\mathcal{F} \ni A \mapsto \phi(A) := \sum_{a\in A} 2^a$$

. It is clear that ϕ increases if A gets bigger: $A \subset B \implies \phi(A) \leqslant \phi(B)$. Let $A, B \in \mathcal{F}$ be two finite sets, say $A = \{a_1, a_2, \dots, a_M\}$ and $\{b_1, b_2, \dots, b_N\}$ (ordered according to size with a_1, b_1 being the smallest and a_M, b_N the biggest) such that $\phi(A) = \phi(B)$. Assume, to the contrary, that $A \neq B$. If $a_M \neq b_N$, say $a_M > b_N$, then

$$\begin{aligned}\phi(A) \geqslant \phi(\{a_M\}) \geqslant 2^{a_M} > \frac{2^{a_M}-1}{2-1} &= \sum_{j=1}^{a_M-1} 2^j \\ &= \phi(\{1, 2, 3, \dots a_M - 1\}) \\ &\geqslant \phi(B),\end{aligned}$$

which cannot be the case since we assumed $\phi(A) = \phi(B)$. Thus, $a_M = b_N$. Now consider recursively the next elements, a_{M-1} and b_{N-1} and the same conclusion yields their equality etc. The process stops after $\min\{M, N\}$ steps. But if $M \neq N$, say $M > N$, then A would contain at least one more element than B, hence $\phi(A) > \phi(B)$, which is also a contradiction. This, finally shows that $A = B$, hence that ϕ is injective.

On the other hand, each natural number can be expressed in terms of finite sums of powers of base-2, so that ϕ is also surjective.

Thus, $\#\mathcal{F} = \#\mathbb{N}$.

Problem 2.20 (Let $\mathcal{F}$ be as in the previous exercise.) Observe that the infinite sets from $\mathcal{P}(\mathbb{N})$, $\mathcal{I} := \mathcal{P}(\mathbb{N}) \setminus \mathcal{F}$ can be surjectively mapped onto $\{0, 1\}^{\mathbb{N}}$: if $\{a_1, a_2, a_3, \dots\} = A \subset \mathbb{N}$, then define an infinite 0-1-sequence $(b_1, b_2, b_3, \dots)$ by setting $b_j = 0$ or $b_j = 1$ according to whether a_j is even or odd. This is a surjection of $\mathcal{P}(\mathbb{N})$ onto $\{0, 1\}^{\mathbb{N}}$ and so $\#\mathcal{P}(\mathbb{N}) \geqslant \#\{0, 1\}^{\mathbb{N}}$. Call this map γ and consider the family $\gamma^{-1}(s)$, $s \in \{0, 1\}^{\mathbb{N}}$ in $\mathcal{I}$, consisting of obviously disjoint infinite subsets of $\mathbb{N}$ which lead to the same 0-1-sequence s. Now choose from each family $\gamma^{-1}(s)$ a representative, call it $r(s) \in \mathcal{I}$. Then the map $s \mapsto r(s)$ is a bijection between $\{0, 1\}^{\mathbb{N}}$ and a subset of $\mathcal{I}$, the set of all representatives. Hence, $\mathcal{I}$ has at least the same cardinality as $\{0, 1\}^{\mathbb{N}}$ and as such a bigger cardinality than $\mathbb{N}$.

Problem 2.21 Denote by Θ the map $\mathcal{P}(\mathbb{N}) \ni A \mapsto \mathbf{1}_A \in \{0, 1\}^{\mathbb{N}}$. Let $\delta = (d_1, d_2, d_3, \dots) \in \{0, 1\}^{\mathbb{N}}$ and define $A(\delta) := \{j \in \mathbb{N} : d_j = 1\}$. Then $\delta = (\mathbf{1}_{A(\delta)}(j))_{j \in \mathbb{N}}$ showing that Θ is surjective.

On the other hand,

$$\mathbf{1}_A = \mathbf{1}_B \iff \mathbf{1}_A(j) = \mathbf{1}_B(j) \ \ \forall j \in \mathbb{N} \iff A = B.$$

This shows the injectivity of Θ, and $\#\mathcal{P}(\mathbb{N}) = \#\{0, 1\}^{\mathbb{N}}$ follows.