11 Convergence theorems and their applications. Solutions to Problems 11.1–11.21

Problem 11.1 We start with the simple remark that

$$\begin{aligned}|a-b|^p &\leqslant (|a|+|b|)^p \\ &\leqslant (\max\{|a|,|b|\}+\max\{|a|,|b|\})^p \\ &= 2^p \max\{|a|,|b|\}^p \\ &= 2^p \max\{|a|^p,|b|^p\} \\ &\leqslant 2^p(|a|^p+|b|^p).\end{aligned}$$

Because of this we find that $|u_j - u|^p \leqslant 2^p g^p$ and the right-hand side is an integrable dominating function.

Proof alternative 1: Apply Theorem 11.2 on dominated convergence to the sequence $\phi_j := |u_j - u|^p$ of integrable functions. Note that $\phi_j(x) \to 0$ and that $0 \leqslant \phi_j \leqslant \Phi$ where $\Phi = 2^p g^p$ is integrable and independent of j. Thus,

$$\begin{aligned}\lim_{j\to\infty}\int |u_j - u|^p\,d\mu = \lim_{j\to\infty}\int \phi_j\,d\mu &= \int \lim_{j\to\infty}\phi_j\,d\mu \\ &= \int 0\,d\mu \;=\; 0.\end{aligned}$$

Proof alternative 2: Mimic the proof of Theorem 11.2 on dominated convergence. To do so we remark that the sequence of functions

$$0 \leqslant \psi_j := 2^p g^p - |u_j - u|^p \xrightarrow{j\to\infty} 2^p g^p$$

Since the limit $\lim_j \psi_j$ exists, it coincides with $\liminf_j \psi_j$, and so we can use Fatou's Lemma to get

$$\begin{aligned}\int 2^p g^p\,d\mu &= \int \liminf_{j\to\infty}\psi_j\,d\mu \\ &\leqslant \liminf_{j\to\infty}\int \psi_j\,d\mu\end{aligned}$$

$$\begin{aligned}
&= \liminf_{j\to\infty} \int \left(2^p g^p - |u_j - u|^p\right) d\mu \\
&= \int 2^p g^p \, d\mu + \liminf_{j\to\infty} \left(- \int |u_j - u|^p \, d\mu \right) \\
&= \int 2^p g^p \, d\mu - \limsup_{j\to\infty} \int |u_j - u|^p \, d\mu
\end{aligned}$$

where we used that $\liminf_j(-\alpha_j) = -\limsup_j \alpha_j$. This shows that $\limsup_j \int |u_j - u|^p \, d\mu = 0$, hence

$$0 \leqslant \liminf_{j\to\infty} \int |u_j - u|^p \, d\mu \leqslant \limsup_{j\to\infty} \int |u_j - u|^p \, d\mu \leqslant 0$$

showing that lower and upper limit coincide and equal to 0, hence $\lim_j \int |u_j - u|^p \, d\mu = 0$.

Problem 11.2 Assume that, as in the statement of Theorem 11.2, $u_j \to u$ and that $|u_j| \leqslant f \in \mathcal{L}^1(\mu)$. In particular,

$$-f \leqslant u_j \quad \text{and} \quad u_j \leqslant f$$

$(j \in \mathbb{N})$ is an integrable minorant resp. majorant. Thus, using Problem 10.8 at $*$ below,

$$\begin{aligned}
\int u \, d\mu &= \int \liminf_{j\to\infty} u_j \, d\mu \\
&\overset{*}{\leqslant} \liminf_{j\to\infty} \int u_j \, d\mu \\
&\leqslant \limsup_{j\to\infty} \int u_j \, d\mu \\
&\overset{*}{\leqslant} \int \limsup_{j\to\infty} u_j \, d\mu \;=\; \int u \, d\mu.
\end{aligned}$$

This proves $\int u \, d\mu = \lim_j \int u_j \, d\mu$.

Addition: since $0 \leqslant |u - u_j| \leqslant |\lim_j u_j| + |u_j| \leqslant 2f \in \mathcal{L}^1(\mu)$, the sequence $|u - u_j|$ has an integrable majorant and using Problem 10.8 we get

$$0 \leqslant \limsup_{j\to\infty} \int |u_j - u| \, d\mu \leqslant \int \limsup_{j\to\infty} |u_j - u| \, d\mu = \int 0 \, d\mu = 0$$

and also (i) of Theorem 11.2 follows...

Problem 11.3 By assumption we have

$$0 \leqslant f_k - g_k \xrightarrow{k\to\infty} f - g,$$
$$0 \leqslant G_k - f_k \xrightarrow{k\to\infty} G - f.$$

Using Fatou's Lemma we find

$$\begin{aligned}\int (f-g)\,d\mu &= \int \lim_k (f_k - g_k)\,d\mu \\ &= \int \liminf_k (f_k - g_k)\,d\mu \\ &\leqslant \liminf_k \int (f_k - g_k)\,d\mu \\ &= \liminf_k \int f_k\,d\mu - \int g\,d\mu,\end{aligned}$$

and

$$\begin{aligned}\int (G-f)\,d\mu &= \int \lim_k (G_k - f_k)\,d\mu \\ &= \int \liminf_k (G_k - f_k)\,d\mu \\ &\leqslant \liminf_k \int (G_k - f_k)\,d\mu \\ &= \int G\,d\mu - \limsup_k \int f_k\,d\mu.\end{aligned}$$

Adding resp. subtracting $\int g\,d\mu$ resp. $\int G\,d\mu$ therefore yields

$$\limsup_k \int f_k\,d\mu \leqslant \int f\,d\mu \leqslant \liminf_k \int f_k\,d\mu$$

and the claim follows.

Problem 11.4 Using Beppo Levi's theorem in the form of Corollary 9.9 we find

$$\int \sum_{j=1}^{\infty} |u_j|\,d\mu = \sum_{j=1}^{\infty} \int |u_j|\,d\mu < \infty, \tag{*}$$

which means that the positive function $\sum_{j=1}^{\infty} |u_j|$ is finite almost everywhere, i.e. the series $\sum_{j=1}^{\infty} u_j$ converges (absolutely) almost everywhere.

Moreover,

$$\int \sum_{j=1}^{N} u_j \, d\mu = \sum_{j=1}^{N} \int u_j \, d\mu \tag{**}$$

and, using the triangle inequality both quantities

$$\Big| \int \sum_{j=n}^{N} u_j \, d\mu \Big| \quad \text{and} \quad \Big| \sum_{j=n}^{N} \int u_j \, d\mu \Big|$$

can be estimated by

$$\int \sum_{j=n}^{N} |u_j| \, d\mu \xrightarrow{n,N\to\infty} 0$$

because of (*). This shows that both sides in (**) are Cauchy sequences, i.e. they are convergent.

Problem 11.5 Since $\mathcal{L}^1(\mu) \ni u_j \downarrow 0$ we find by monotone convergence, Theorem 11.1, that $\int u_j \, d\mu \downarrow 0$. Therefore,

$$\sigma = \sum_{j=1}^{\infty} (-1)^j u_j \quad \text{and} \quad S = \sum_{j=1}^{\infty} (-1)^j \int u_j \, d\mu \quad \text{converge}$$

(conditionally, in general). Moreover, for every $N \in \mathbb{N}$,

$$\int \sum_{j=1}^{N} (-1)^j u_j \, d\mu = \sum_{j=1}^{N} \int (-1)^j u_j \, d\mu \xrightarrow{N\to\infty} S.$$

All that remains is to show that the right-hand side converges to $\int \sigma \, d\mu$. Observe that for $S_N := \sum_{j=1}^{N} (-1)^j u_j$ we have

$$S_{2N} \leqslant S_{2N+2} \leqslant \ldots \leqslant S$$

and we find, as $S_j \in \mathcal{L}^1(\mu)$, by monotone convergence that

$$\lim_{N\to\infty} \int S_{2N} \, d\mu = \int \sigma \, d\mu.$$

Problem 11.6 Consider $u_j(x) := j \cdot \mathbf{1}_{(0,1/j)}(x)$, $j \in \mathbb{N}$. It is clear that u_j is measurable and Lebesgue integrable with integral

$$\int u_j \, d\lambda = j \, \frac{1}{j} = 1 \qquad \forall \, j \in \mathbb{N}.$$

Thus, $\lim_j \int u_j \, d\lambda = 1$. On the other hand, the pointwise limit is

$$u(x) := \lim_j u_j(x) \equiv 0$$

so that $0 = \int u \, d\lambda = \int \lim_j u_j \, d\lambda \neq 1$.

The example does not contradict dominated convergence as there is no uniform dominating integrable function.

Alternative: a similar situation can be found for $v_k(x) := \frac{1}{k} \mathbf{1}_{[0,k]}(x)$ and the pointwise limit $v \equiv 0$. Note that in this case the limit is even uniform and still $\lim_k \int v_k \, d\lambda = 1 \neq 0 = \int v \, d\lambda$. Again there is no contradiction to dominated convergence as there does not exist a uniform dominating integrable function.

Problem 11.7 Let μ be an arbitrary Borel measure on the line $\mathbb{R}$ and define the integral function for some $u \in \mathcal{L}^1(\mu)$ through

$$I(x) := I_\mu^u(x) := \int_{(0,x)} u(t) \, \mu(dt) = \int \mathbf{1}_{(0,x)}(t) u(t) \, \mu(dt).$$

For any sequence $0 < l_j \to x$, $l_j < x$ from the left and $r_k \to x$, $r_k > x$ from the right we find

$$\mathbf{1}_{(0,l_j)}(t) \xrightarrow{j\to\infty} \mathbf{1}_{(0,x)}(t) \quad \text{and} \quad \mathbf{1}_{(0,r_k)}(t) \xrightarrow{k\to\infty} \mathbf{1}_{(0,x]}(t).$$

Since $|\mathbf{1}_{(0,x)} u| \leqslant |u| \in \mathcal{L}^1$ is a uniform dominating function, Lebesgue's dominated convergence theorem yields

$$\begin{aligned}
I(x+) - I(x-) &= \lim_k I(r_k) - \lim_j I(l_j) \\
&= \int \mathbf{1}_{(0,x]}(t) u(t) \, \mu(dt) - \int \mathbf{1}_{(0,x)}(t) u(t) \, \mu(dt) \\
&= \int \big(\mathbf{1}_{(0,x]}(t) - \mathbf{1}_{(0,x)}(t) \big) u(t) \, \mu(dt) \\
&= \int \mathbf{1}_{\{x\}}(t) u(t) \, \mu(dt) \\
&= u(x) \, \mu(\{x\}).
\end{aligned}$$

Thus $I(x)$ is continuous at x if, and only if, x is not an atom of μ.

Remark: the proof shows, by the way, that $I_\mu^u(x)$ is *always* left-continous at every x, no matter what μ or u look like.

Problem 11.8 (i) We have

$$\begin{aligned}
&\int \frac{1}{x}\,\mathbf{1}_{[1,\infty)}(x)\,dx \\
&\qquad = \lim_{n\to\infty} \int \frac{1}{x}\,\mathbf{1}_{[1,n)}(x)\,dx && \text{by Beppo Levi's thm.} \\
&\qquad = \lim_{n\to\infty} \int_{[1,n)} \frac{1}{x}\,dx && \text{usual shorthand} \\
&\qquad = \lim_{n\to\infty} (R)\int_1^n \frac{1}{x}\,dx && \text{Riemann-}\int_1^n \text{ exists} \\
&\qquad = \lim_{n\to\infty} [\log x]_1^n \\
&\qquad = \lim_{n\to\infty} [\log(n) - \log(1)] = \infty
\end{aligned}$$

which means that $\frac{1}{x}$ is not Lebesgue-integrable over $[1,\infty)$.

(ii) We have

$$\begin{aligned}
&\int \frac{1}{x^2}\,\mathbf{1}_{[1,\infty)}(x)\,dx \\
&\qquad = \lim_{n\to\infty} \int \frac{1}{x^2}\,\mathbf{1}_{[1,n)}(x)\,dx && \text{by Beppo Levi's thm.} \\
&\qquad = \lim_{n\to\infty} \int_{[1,n)} \frac{1}{x^2}\,dx && \text{usual shorthand} \\
&\qquad = \lim_{n\to\infty} (R)\int_1^n \frac{1}{x^2}\,dx && \text{Riemann-}\int_1^n \text{ exists} \\
&\qquad = \lim_{n\to\infty} \left[-\tfrac{1}{x}\right]_1^n \\
&\qquad = \lim_{n\to\infty} [1 - \tfrac{1}{n}] = 1 < \infty
\end{aligned}$$

which means that $\frac{1}{x^2}$ is Lebesgue-integrable over $[1,\infty)$.

(iii) We have

$$\begin{aligned}
&\int \frac{1}{\sqrt{x}}\,\mathbf{1}_{(0,1]}(x)\,dx \\
&\qquad = \lim_{n\to\infty} \int \frac{1}{\sqrt{x}}\,\mathbf{1}_{(1/n,1]}(x)\,dx && \text{by Beppo Levi's thm.} \\
&\qquad = \lim_{n\to\infty} \int_{(1/n,1]} \frac{1}{\sqrt{x}}\,dx && \text{usual shorthand} \\
&\qquad = \lim_{n\to\infty} (R)\int_{1/n}^1 \frac{1}{\sqrt{x}}\,dx && \text{Riemann-}\int_{1/n}^1 \text{ exists}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \left[2\sqrt{x}\right]_{1/n}^{1} \\
&= \lim_{n\to\infty} \left[2 - 2\sqrt{\tfrac{1}{n}}\right] \\
&= 2 < \infty
\end{aligned}
$$

which means that $\frac{1}{\sqrt{x}}$ is Lebesgue-integrable over $(0,1]$.

(iv) We have

$$
\begin{aligned}
&\int \frac{1}{x}\,\mathbf{1}_{(0,1]}(x)\,dx \\
&\qquad = \lim_{n\to\infty} \int \frac{1}{x}\,\mathbf{1}_{(1/n,1]}(x)\,dx && \text{by Beppo Levi's thm.} \\
&\qquad = \lim_{n\to\infty} \int_{(1/n,1]} \frac{1}{x}\,dx && \text{usual shorthand} \\
&\qquad = \lim_{n\to\infty} (R)\int_{1/n}^{1} \frac{1}{x}\,dx && \text{Riemann-}\int_{1/n}^{1} \text{ exists} \\
&\qquad = \lim_{n\to\infty} \left[\log x\right]_{1/n}^{1} \\
&\qquad = \lim_{n\to\infty} \left[\log(1) - \log\tfrac{1}{n}\right] \\
&\qquad = \infty
\end{aligned}
$$

which means that $\frac{1}{x}$ is not Lebesgue-integrable over $(0,1]$.

Problem 11.9 We construct a dominating integrable function.

If $x \leqslant 1$, we have clearly $\exp(-x^\alpha) \leqslant 1$, and $\int_{(0,1]} \mathbf{1}\,dx = 1 < \infty$ is integrable.

If $x \geqslant 1$, we have $\exp(-x^\alpha) \leqslant Mx^{-2}$ for some suitable constant $M = M_\alpha < \infty$. This function is integrable in $[1,\infty)$, see e.g. Problem 11.8. The estimate is easily seen from the fact that $x \mapsto x^2\exp(-x^\alpha)$ is continuous in $[1,\infty)$ with $\lim_{x\to\infty} x^2\exp(-x^\alpha) = 0$.

This shows that $\exp(-x^\alpha) \leqslant \mathbf{1}_{(0,1)} + Mx^{-2}\,\mathbf{1}_{[1,\infty)}$ with the right-hand side being integrable.

Problem 11.10 Take $\alpha \in (a,b)$ where $0 < a < b < \infty$ are fixed (but arbitrary). We show that the function is continuous for these α. This shows the general case since continuity is a local property and we can 'catch' any given α_0 by some choice of a and b's.

We use the Continuity lemma (Theorem 11.4) and have to find uniform (for $\alpha \in (a,b)$) dominating bounds on the integrand function

$f(\alpha,x) := \left(\frac{\sin x}{x}\right)^3 e^{-\alpha x}$. First of all, we remark that $\left|\frac{\sin x}{x}\right| \leqslant M$ which follows from the fact that $\frac{\sin x}{x}$ is a continuous function such that $\lim_{x\to\infty}\frac{\sin x}{x} = 0$ and $\lim_{x\downarrow 0}\frac{\sin x}{x} = 1$. (Actually, we could choose $M = 1$...). Moreover, $\exp(-\alpha x) \leqslant 1$ for $x \in (0,1)$ and $\exp(-\alpha x) \leqslant C_{a,b}x^{-2}$ for $x \geqslant 1$—use for this the continuity of $x^2\exp(-\alpha x)$ and the fact that $\lim_{x\to\infty} x^2\exp(-\alpha x) = 0$. This shows that

$$|f(\alpha,x)| \leqslant M\left(\mathbf{1}_{(0,1)}(x) + C_{a,b}x^{-2}\,\mathbf{1}_{[1,\infty)}(x)\right)$$

and the right-hand side is an integrable dominating function which does not depend on α—as long as $\alpha \in (a,b)$. But since $\alpha \mapsto f(\alpha,x)$ is obviously continuous, the Continuity lemma applies and proves that $\int_{(0,\infty)} f(\alpha,x)\,dx$ is continuous.

Problem 11.11 Fix some number $N > 0$ and take $x \in (-N,N)$. We show that $G(x)$ is continuous on this set. Since N was arbitrary, we find that G is continuous for every $x \in \mathbb{R}$.

Set $g(t,x) := \frac{\sin(tx)}{t(1+t^2)} = x\,\frac{\sin(tx)}{(tx)}\,\frac{1}{1+t^2}$. Then, using that $\left|\frac{\sin u}{u}\right| \leqslant M$, we have

$$|g(t,x)| \leqslant x\cdot M\cdot\frac{1}{1+t^2} \leqslant M\cdot N\cdot\left(\mathbf{1}_{(0,1)}(t) + \frac{1}{t^2}\,\mathbf{1}_{[1,\infty)}(t)\right)$$

and the right-hand side is a uniformly dominating function, i.e. $G(x)$ makes sense and we find $G(0) = \int_{t\neq 0} g(t,0)\,dt = 0$. To see differentiability, we use the Differentiability lemma (Theorem 11.5) and need to prove that $|\partial_x g(t,x)|$ exists (this is clear) and is uniformly dominated for $x \in (-N,N)$. We have

$$\begin{aligned}|\partial_x g(t,x)| = \left|\partial_x\frac{\sin(tx)}{t(1+t^2)}\right| &= \left|\frac{\cos(tx)}{(1+t^2)}\right| \\ &\leqslant \frac{1}{1+t^2} \\ &\leqslant \left(\mathbf{1}_{(0,1)}(t) + \frac{1}{t^2}\,\mathbf{1}_{[1,\infty)}(t)\right)\end{aligned}$$

and this allows us to apply the Differentiability lemma, so

$$\begin{aligned}G'(x) = \partial_x\int_{t\neq 0} g(t,x)\,dt &= \int_{t\neq 0}\partial_x g(t,x)\,dt \\ &= \int_{t\neq 0}\frac{\cos(tx)}{1+t^2}\,dt\end{aligned}$$

$$= \int_{\mathbb{R}} \frac{\cos(tx)}{1+t^2}\,dt$$

(use in the last equality that $\{0\}$ is a Lebesgue null set). Thus, by a Beppo-Levi argument (and using that Riemann=Lebesgue whenever the Riemann integral over a compact interval exists...)

$$\begin{aligned} G'(0) = \int_{\mathbb{R}} \frac{1}{1+t^2}\,dt &= \lim_{n\to\infty} (R)\int_{-n}^{n} \frac{1}{1+t^2}\,dt \\ &= \lim_{n\to\infty} [\tan^{-1}(t)]_{-n}^{n} \\ &= \pi. \end{aligned}$$

Now observe that

$$\partial_x \sin(tx) = t\cos(tx) = \frac{t}{x}\,x\cos(tx) = \frac{t}{x}\,\partial_t \sin(tx).$$

Since the integral defining $G'(x)$ exists we can use a Beppo-Levi argument, Riemann=Lebesgue (whenever the Riemann integral over an interval exists) and integration by parts (for the Riemann integral) to find

$$\begin{aligned} xG'(x) &= \int_{\mathbb{R}} \frac{x\cos(tx)}{1+t^2}\,dt \\ &= \lim_{n\to\infty} (R)\int_{-n}^{n} \frac{x\partial_x \sin(tx)}{t(1+t^2)}\,dt \\ &= \lim_{n\to\infty} (R)\int_{-n}^{n} \frac{t\partial_t \sin(tx)}{t(1+t^2)}\,dt \\ &= \lim_{n\to\infty} (R)\int_{-n}^{n} \frac{\partial_t \sin(tx)}{1+t^2}\,dt \\ &= \lim_{n\to\infty} (R)\int_{-n}^{n} \partial_t \sin(tx)\cdot\frac{1}{1+t^2}\,dt \\ &= \lim_{n\to\infty}\left[\frac{\sin(tx)}{1+t^2}\right]_{t=-n}^{n} - \lim_{n\to\infty} (R)\int_{-n}^{n} \sin(tx)\cdot\partial_t\frac{1}{1+t^2}\,dt \\ &= \lim_{n\to\infty} (R)\int_{-n}^{n} \sin(tx)\cdot\frac{2t}{(1+t^2)^2}\,dt \\ &= \int_{\mathbb{R}} \frac{2t\sin(tx)}{(1+t^2)^2}\,dt. \end{aligned}$$

Problem 11.12 (i) Note that for $0 \leqslant a, b \leqslant 1$

$$1-(1-a)^b = \int_{1-a}^{1} bt^{b-1}\,dt \geqslant \int_{1-a}^{1} b\,dt = ba$$

so that we get for $0 \leqslant x \leqslant k$ and $a := x/k$, $b := k/(k+1)$

$$\Big(1 - \frac{x}{k}\Big)^{\frac{k}{k+1}} \leqslant 1 - \frac{x}{k+1}, \qquad 0 \leqslant x \leqslant k$$

or,

$$\Big(1 - \frac{x}{k}\Big)^{k} \mathbf{1}_{[0,k]}(x) \leqslant \Big(1 - \frac{x}{k+1}\Big)^{k+1} \mathbf{1}_{[0,k+1]}(x).$$

Therefore we can appeal to Beppo Levi's theorem to get

$$\begin{aligned}
\lim_{k\to\infty} \int_{(1,k)} \Big(1 - \frac{x}{k}\Big)^{k} \ln x \,\lambda^1(dx) &= \sup_{k\in\mathbb{N}} \int \mathbf{1}_{(1,k)}(x)\Big(1 - \frac{x}{k}\Big)^{k} \ln x \,\lambda^1(dx) \\
&= \int \sup_{k\in\mathbb{N}} \Big[\mathbf{1}_{(1,k)}(x)\Big(1 - \frac{x}{k}\Big)^{k}\Big] \ln x \,\lambda^1(dx) \\
&= \int \mathbf{1}_{(1,\infty)}(x) e^{-x} \ln x \,\lambda^1(dx).
\end{aligned}$$

That $e^{-x}\ln x$ is integrable in $(1,\infty)$ follows easily from the estimates

$$e^{-x} \leqslant C_N x^{-N} \quad \text{and} \quad \ln x \leqslant x$$

which hold for all $x \geqslant 1$ and $N \in \mathbb{N}$.

(ii) Note that $x \mapsto \ln x$ is continuous and bounded in $[\epsilon, 1]$, thus Riemann integrable. It is easy to see that $x\ln x - x$ is a primitive for $\ln x$. The improper Riemann integral

$$\int_0^1 \ln x \,dx = \lim_{\epsilon\to 0} [x\ln x - x]_\epsilon^1 = -1$$

exists and, since $\ln x$ is negative throughout $(0,1)$, improper Riemann and Lebesgue integrals coincide. Thus, $\ln x \in L^1(dx, (0,1))$. Therefore,

$$\left|\Big(1 - \frac{x}{k}\Big)^{k} \ln x\right| \leqslant |\ln x|, \quad \forall\, x \in (0,1)$$

is uniformly dominated by an integrable function and we can use dominated convergence to get

$$\begin{aligned}
\lim_{k} \int_{(0,1)} \Big(1 - \frac{x}{k}\Big)^{k} \ln x \,dx &= \int_{(0,1)} \lim_{k} \Big(1 - \frac{x}{k}\Big)^{k} \ln x \,dx \\
&= \int_{(0,1)} e^{-x} \ln x \,dx
\end{aligned}$$

Problem 11.13 Fix throughout $(a,b)\subset(0,\infty)$ and take $x\in(a,b)$. Let us remark that, just as in Problem 11.8, we prove that

$$\int_{(0,1)} t^{-\delta}\,dt<\infty \quad \forall\,\delta<1 \qquad \text{and} \qquad \int_{(1,\infty)} t^{-\delta}\,dt<\infty \quad \forall\,\delta>1.$$

(i) That the integrand function $x\mapsto\gamma(t,x)$ is continuous on (a,b) is clear. It is therefore enough to find an integrable dominating function. We have

$$e^{-t}t^{x-1}\leqslant t^{a-1} \qquad \forall\,t\in(0,1),\quad x\in(a,b)$$

which is clearly integrable on $(0,1)$ and

$$e^{-t}t^{x-1}\leqslant M_{a,b}\,t^{-2} \qquad \forall\,t\geqslant 1,\quad x\in(a,b)$$

where we used that $t^\rho e^{-t}$, $\rho>0$, is continuous and $\lim_{t\to\infty} t^\rho e^{-t}=0$ to find $M_{a,b}$. This function is integrable over $[1,\infty)$. Both estimates together give the wanted integrable dominating function. The Continuity lemma (Theorem 11.4) applies. The well-definedness of $\Gamma(x)$ comes for free as a by-product of the existence of the dominating function.

(ii) *Induction Hypothesis:* $\Gamma^{(m)}$ exists and is of the form as claimed in the statement of the problem.

Induction Start $m=1$: We have to show that $\Gamma(x)$ is differentiable. We want to use the Differentiability lemma, Theorem 11.5. For this we remark first of all, that the integrand function $x\mapsto\gamma(t,x)$ is differentiable on (a,b) and that

$$\partial_x\gamma(t,x)=\partial_x e^{-t}\,t^{x-1}=e^{-t}\,t^{x-1}\,\log t.$$

We have now to find a uniform (for $x\in(a,b)$) integrable dominating function for $|\partial_x\gamma(t,x)|$. Since $\log t\leqslant t$ for all $t>0$ (the logarithm is a concave function!),

$$\begin{aligned}\left|e^{-t}\,t^{x-1}\,\log t\right| &= e^{-t}\,t^{x-1}\,\log t\\ &\leqslant e^{-t}t^{x} \;\leqslant\; e^{-t}t^{b} \;\leqslant\; C_b\,t^{-2} \qquad \forall\,t\geqslant 1,\quad x\in(a,b)\end{aligned}$$

(use for the last step the argument used in part (i) of this problem). Moreover,

$$\left|e^{-t}\,t^{x-1}\,\log t\right|\leqslant t^{a-1}\,|\log t|$$

$$= t^{a-1} \log \frac{1}{t} \leqslant C_a\, t^{-1/2} \qquad \forall\, t \in (0,1), \quad x \in (a,b)$$

where we used the fact that $\lim_{t\to 0} t^\rho \log \frac{1}{t} = 0$ which is easily seen by the substitution $t = e^{-u}$ and $u \to \infty$ and the continuity of the function $t^\rho \log \frac{1}{t}$.

Both estimates together furnish an integrable dominating function, so the Differentiability lemma applies and shows that

$$\Gamma'(x) = \int_{(0,\infty)} \partial_x \gamma(t,x)\, dt = \int_{(0,\infty)} e^{-t}\, t^{x-1} \log t\, dt = \Gamma^{(1)}(x).$$

Induction Step $m \rightsquigarrow m+1$: Set $\gamma^{(m)}(t,x) = e^{-t}\, t^{x-1} (\log t)^m$. We want to apply the Differentiability Lemma to $\Gamma^{(m)}(x)$. With very much the same arguments as in the induction start we find that $\gamma^{(m+1)}(t,x) = \partial_x \gamma^{(m)}(t,x)$ exists (obvious) and satisfies the following bounds

$$\begin{aligned}
\left|e^{-t}\, t^{x-1} (\log t)^{m+1}\right| &= e^{-t}\, t^{x-1} (\log t)^{m+1} \\
&\leqslant e^{-t} t^{x+m} \\
&\leqslant e^{-t} t^{b+m} \\
&\leqslant C_{b,m}\, t^{-2} \qquad \forall\, t \geqslant 1, \quad x \in (a,b) \\
\left|e^{-t}\, t^{x-1} (\log t)^{m+1}\right| &\leqslant t^{a-1}\, |\log t|^m \\
&= t^{a-1} \left(\log \frac{1}{t}\right)^{m+1} \\
&\leqslant C_{a,m}\, t^{-1/2} \qquad \forall\, t \in (0,1), \quad x \in (a,b)
\end{aligned}$$

and the Differentiability lemma applies completing the induction step.

(iii) Using a combination of Beppo-Levi (indicated by 'B-L'), Riemann=Lebesgue (if the Riemann integral over an interval exists) and integration by parts (for the Riemann integral, indicated by 'I-by-P') techniques we get

$$\begin{aligned}
x\Gamma(x) &= \lim_{n\to\infty} \int_{(1/n,n)} e^{-t}\, x t^{x-1}\, dt && \text{B-L} \\
&= \lim_{n\to\infty} (R) \int_{1/n}^{n} e^{-t}\, \partial_t t^x\, dt \\
&= \lim_{n\to\infty} \left[e^{-t}\, t^x\right]_{t=1/n}^{n} - \lim_{n\to\infty} (R) \int_{1/n}^{n} \partial_t e^{-t}\, t^x\, dt && \text{I-by-P}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} (R) \int_{1/n}^{n} e^{-t}\, t^{(x+1)-1}\, dt \\
&= \lim_{n\to\infty} \int_{(1/n,n)} e^{-t}\, t^{(x+1)-1}\, dt \\
&= \int_{(0,\infty)} e^{-t}\, t^{(x+1)-1}\, dt && \text{B-L} \\
&= \Gamma(x+1).
\end{aligned}
$$

Problem 11.14 Fix $(a,b) \subset (0,1)$ and let always $u \in (a,b)$. We have for $x \geqslant 0$ and $L \in \mathbb{N}_0$

$$
\begin{aligned}
|x^L f(u,x)| &= |x|^L \left| \frac{e^{ux}}{e^x + 1} \right| \\
&= x^L \, \frac{e^{ux}}{e^x + 1} \\
&\leqslant x^L \, \frac{e^{ux}}{e^x} \\
&= x^L \, e^{(u-1)x} \\
&\leqslant \mathbf{1}_{[0,1]}(x) + M_{a,b} \mathbf{1}_{(1,\infty)}(x)\, x^{-2}
\end{aligned}
$$

where we used that $u-1 < 0$, the continuity and boundedness of $x^\rho e^{-ax}$ for $x \in [1,\infty)$ and $\rho \geqslant 0$. If $x \leqslant 0$ we get

$$
\begin{aligned}
|x^L f(u,x)| &= |x|^L \left| \frac{e^{ux}}{e^x + 1} \right| \\
&= |x|^L \, e^{-u|x|} \\
&\leqslant \mathbf{1}_{[-1,0]}(x) + N_{a,b} \mathbf{1}_{(-\infty,1)}(x)\, |x|^{-2}.
\end{aligned}
$$

Both inequalities give dominating functions which are integrable; therefore, the integral $\int_{\mathbb{R}} x^L f(u,x)\, dx$ exists.

To see m-fold differentiability, we use the Differentiability lemma (Theorem 11.5) m-times. Formally, we have to use induction. Let us only make the induction step (the start is very similar!). For this, observe that

$$
\partial_u^m (x^n f(u,x)) = \partial_u^m \frac{x^n e^{ux}}{e^x + 1} = \frac{x^{n+m} e^{ux}}{e^x + 1}
$$

but, as we have seen in the first step with $L = n+m$, this is uniformly bounded by an integrable function. Therefore, the Differentiability lemma applies and shows that

$$
\partial_u^m \int_{\mathbb{R}} x^n\, f(u,x)\, dx = \int_{\mathbb{R}} x^n\, \partial_u^m f(u,x)\, dx = \int_{\mathbb{R}} x^{n+m}\, f(u,x)\, dx.
$$

Problem 11.15 Note the *misprint* in this problem: the random variable X should be *positive*.

(i) Since

$$\left|\frac{d^m}{dt^m}\, e^{-tX}\right| = \left|X^m e^{-tX}\right| \leqslant X^m$$

m applications of the differentiability lemma, Theorem 11.5, show that $\phi_X^{(m)}(0+)$ exists and that

$$\phi_X^{(m)}(0+) = (-1)^m \int X^m \, dP.$$

(ii) Using the exponential series we find that

$$\begin{aligned} e^{-tX} - \sum_{k=0}^{m} X^k \frac{(-1)^k t^k}{k!} &= \sum_{k=m+1}^{\infty} X^k \frac{(-1)^k t^k}{k!} \\ &= t^{m+1} \sum_{j=0}^{\infty} X^{m+1+j} \frac{(-1)^{m+1+j} t^j}{(m+1+j)!}. \end{aligned}$$

Since the left-hand side has a finite P-integral, so has the right, i.e.

$$\int \Big(\sum_{j=0}^{\infty} X^{m+1+j} \frac{(-1)^{m+1+j} t^j}{(m+1+j)!} \Big) \, dP \qquad \text{converges}$$

and we see that

$$\int \Big(e^{-tX} - \sum_{k=0}^{m} X^k \frac{(-1)^k t^k}{k!} \Big) \, dP = o(t^m)$$

as $t \to 0$.

(iii) We show, by induction in m, that

$$\left| e^{-u} - \sum_{k=0}^{m-1} \frac{(-u)^k}{k!} \right| \leqslant \frac{u^m}{m!} \qquad \forall\, u \geqslant 0. \tag{*}$$

Because of the elementary inequality

$$|e^{-u} - 1| \leqslant u \qquad \forall\, u \geqslant 0$$

the start of the induction $m = 1$ is clear. For the induction step $m \to m+1$ we note that

$$\left| e^{-u} - \sum_{k=0}^{m} \frac{(-u)^k}{k!} \right| = \left| \int_0^u \Big(e^{-y} - \sum_{k=0}^{m-1} \frac{(-y)^k}{k!} \Big) \, dy \right|$$

$$
\begin{aligned}
&\leqslant \int_0^u \left| e^{-y} - \sum_{k=0}^{m-1} \frac{(-y)^k}{k!} \right| dy \\
&\overset{(*)}{\leqslant} \int_0^u \frac{y^m}{m!}\, dy \\
&= \frac{u^{m+1}}{(m+1)!}\,,
\end{aligned}
$$

and the claim follows.

Setting $x = tX$ in (*), we find by integration that

$$
\pm\left(\int e^{-tX} - \sum_{k=0}^{m-1} (-1)^k t^k \, \frac{\int X^k \, dP}{k!} \right) \leqslant \frac{t^m \int X^m \, dP}{m!}\,.
$$

(iv) If t is in the radius of convergence of the power series, we know that

$$
\lim_{m\to\infty} \frac{|t|^m \int X^m \, dP}{m!} = 0
$$

which, when combined with (iii), proves that

$$
\phi_X(t) = \lim_{m\to\infty} \sum_{k=0}^{m-1} (-1)^k t^k \, \frac{\int X^k \, dP}{k!}\,.
$$

Problem 11.16 (i) Wrong, u is NOT continuous on the irrational numbers. To see this, just take a sequence of rationals $q_j \in \mathbb{Q} \cap [0,1]$ approximating $p \in [0,1] \setminus \mathbb{Q}$. Then

$$
\lim_j u(q_j) = 1 \neq 0 = u(p) = u(\lim_j q_j).
$$

(ii) True. Mind that v is not continuous at 0, but $\{n^{-1}, n \in \mathbb{N}\} \cup \{0\}$ is still countable.

(iii) True. The points where u and v are not 0 (that is: where they are 1) are countable sets, hence measurable and also Lebesgue null sets. This shows that u, v are measurable and almost everywhere 0, hence $\int u \, d\lambda = 0 = \int v \, d\lambda$.

(iv) True. Since $\mathbb{Q} \cap [0,1]$ as well as $[0,1] \setminus \mathbb{Q}$ are dense subsets of $[0,1]$, ALL lower resp. upper Darboux sums are always

$$
S_\pi[u] \equiv 0 \quad \text{resp.} \quad S^\pi[u] \equiv 1
$$

(for any finite partition π of $[0,1]$). Thus upper and lower integrals of u have the value 0 resp. 1 and it follows that u cannot be Riemann integrable.

Problem 11.17 Note that every function which has finitely many discontinuities is Riemann integrable. Thus, if $\{q_j\}_{j\in\mathbb{N}}$ is an enumeration of $\mathbb{Q}$, the functions $u_j(x) := \mathbf{1}_{\{q_1,q_2,\dots,q_j\}}(x)$ are Riemann integrable (with Riemann integral 0) while their increasing limit $u_\infty = \mathbf{1}_{\mathbb{Q}}$ is not Riemann integrable.

Problem 11.18 Of course we have to assume that u is Borel measurable! By assumption we know that $u_j := u\mathbf{1}_{[0,j]}$ is (properly) Riemann integrable, hence Lebesgue integrable and

$$\int_{[0,j]} u\,d\lambda = \int_{[0,j]} u_j\,d\lambda = (\mathrm{R})\int_0^j u(x)\,dx \xrightarrow{j\to\infty} \int_0^\infty u(x)\,dx.$$

The last limit exists because of improper Riemann integrability. Moreover, this limit is an increasing limit, i.e. a 'sup'. Since $0 \leqslant u_j \uparrow u$ we can invoke Beppo Levi's theorem and get

$$\int u\,d\lambda = \sup_j \int u_j\,d\lambda = \int_0^\infty u(x)\,dx < \infty$$

proving Lebesgue integrability.

Problem 11.19 Observe that $x^2 = k\pi \iff x = \sqrt{k\pi},\ x \geqslant 0,\ k \in \mathbb{N}_0$. Thus, Since $\sin x^2$ is continuous, it is on every bounded interval Riemann integrable. By a change of variables, $y = x^2$, we get

$$\int_{\sqrt{a}}^{\sqrt{b}} |\sin(x^2)|\,dx = \int_a^b |\sin y|\,\frac{dy}{2\sqrt{y}} = \int_a^b \frac{|\sin y|}{2\sqrt{y}}\,dy$$

which means that for $a = a_k = k\pi$ and $b = b_k = (k+1)\pi = a_{k+1}$ the values $\int_{\sqrt{a_k}}^{\sqrt{a_{k+1}}} |\sin(x^2)|\,dx$ are a decreasing sequence with limit 0. Since on $\left[\sqrt{a_k}, \sqrt{a_{k+1}}\right]$ the function $\sin x^2$ has only one sign (and alternates its sign from interval to interval), we can use Leibniz' convergence criterion to see that the series

$$\sum_k \int_{\sqrt{a_k}}^{\sqrt{a_{k+1}}} \sin(x^2)\,dx \tag{*}$$

converges, hence the improper integral exists.

The function $\cos x^2$ can be treated similarly. Alternatively, we remark that $\sin x^2 = \cos(x^2 - \pi/2)$.

The functions are not Lebesgue integrable. Either we show that the series (*) does not converge absolutely, or we argue as follows:

$\sin x^2 = \cos(x^2 - \pi/2)$ shows that $\int |\sin x^2|\,dx$ and $\int |\cos x^2|\,dx$ either both converge or diverge. If they would converge (this is equivalent to Lebesgue integrability...) we would find because of $\sin^2 + \cos^2 \equiv 1$ and $|\sin|, |\cos| \leqslant 1$,

$$\begin{aligned}\infty = \int_0^\infty 1\,dx &= \int_0^\infty \left[(\sin x^2)^2 + (\cos x^2)^2\right] dx \\ &= \int_0^\infty (\sin x^2)^2\,dx + \int_0^\infty (\cos x^2)^2\,dx \\ &\leqslant \int_0^\infty |\sin x^2|\,dx + \int_0^\infty |\cos x^2|\,dx \ < \ \infty,\end{aligned}$$

which is a contradiction.

Problem 11.20 Let $r < s$ and, without loss of generality, $a \leqslant b$. A change of variables yields

$$\begin{aligned}\int_r^s \frac{f(bx) - f(ax)}{x}\,dx &= \int_r^s \frac{f(bx)}{x}\,dx - \int_r^s \frac{f(ax)}{x}\,dx \\ &= \int_{br}^{bs} \frac{f(y)}{y}\,dy - \int_{ar}^{as} \frac{f(y)}{y}\,dy \\ &= \int_{as}^{bs} \frac{f(y)}{y}\,dy - \int_{ar}^{br} \frac{f(y)}{y}\,dy\end{aligned}$$

Using the mean value theorem for integrals, E.12, we get

$$\begin{aligned}\int_r^s \frac{f(bx) - f(ax)}{x}\,dx &= f(\xi_s)\int_{as}^{bs} \frac{1}{y}\,dy - f(\xi_r)\int_{ar}^{br} \frac{1}{y}\,dy \\ &= f(\xi_s)\ln\tfrac{b}{a} - f(\xi_r)\ln\tfrac{b}{a}.\end{aligned}$$

Since $\xi_s \in (as, bs)$ and $\xi_r \in (ar, br)$, we find that $\xi_s \xrightarrow{s\to\infty} \infty$ and $\xi_r \xrightarrow{r\to 0} 0$ which means that

$$\int_r^s \frac{f(bx) - f(ax)}{x}\,dx = \left[f(\xi_s) - f(\xi_r)\right]\ln\tfrac{b}{a} \xrightarrow[r\to 0]{s\to\infty} (M - m)\ln\tfrac{b}{a}.$$

Problem 11.21 (i) The function $x \mapsto x\ln x$ is bounded and continuous in $[0, 1]$, hence Riemann integrable. Since in this case Riemann and Lebesgue integrals coincide, we may use Riemann's integral and the usual rules for integration. Thus, changing variables according to $x = e^{-t}$, $dx = -e^{-t}\,dt$ and then $s = (k+1)t$, $ds = (k+1)\,ds$ we find,

$$\begin{aligned}\int_0^1 (x\ln x)^k\,dx &= \int_0^\infty \big[e^{-t}(-t)\big]^k\,e^{-t}\,dt\\ &= (-1)^k\int_0^\infty t^k e^{-t(k+1)}\,dt\\ &= (-1)^k\int_0^\infty \Big(\frac{s}{k+1}\Big)^k e^{-s}\,\frac{ds}{k+1}\\ &= (-1)^k\Big(\frac{1}{k+1}\Big)^{k+1}\int_0^\infty s^{(k+1)-1}e^{-s}\,ds\\ &= (-1)^k\Big(\frac{1}{k+1}\Big)^{k+1}\Gamma(k+1).\end{aligned}$$

(ii) Following the hint we write

$$x^{-x} = e^{-x\ln x} = \sum_{k=0}^\infty (-1)^k\frac{(x\ln x)^k}{k!}.$$

Since for $x \in (0,1)$ the terms under the sum are all positive, we can use Beppo Levi's theorem and the formula $\Gamma(k+1) = k!$ to get

$$\begin{aligned}\int_{(0,1)} x^{-x}\,dx &= \sum_{k=0}^\infty (-1)^k\frac{1}{k!}\int_{(0,1)}(x\ln x)^k\,dx\\ &= \sum_{k=0}^\infty (-1)^k\frac{1}{k!}(-1)^k\Big(\frac{1}{k+1}\Big)^{k+1}\Gamma(k+1)\\ &= \sum_{k=0}^\infty \Big(\frac{1}{k+1}\Big)^{k+1}\\ &= \sum_{n=1}^\infty \Big(\frac{1}{n}\Big)^n.\end{aligned}$$

12 The function spaces $\mathcal{L}^p$, $1 \leqslant p \leqslant \infty$. Solutions to Problems 12.1–12.22

Problem 12.1 (i) We use Hölder's inequality for $r, s \in (1, \infty)$ and $\frac{1}{s} + \frac{1}{t} = 1$ to get

$$\begin{aligned}\|u\|_q^q = \int |u|^q \, d\mu = \int |u|^q \cdot \mathbf{1} \, d\mu \\ &\leqslant \left(\int |u|^{qr} \, d\mu \right)^{1/r} \cdot \left(\int \mathbf{1}^s \, d\mu \right)^{1/s} \\ &= \left(\int |u|^{qr} \, d\mu \right)^{1/r} \cdot (\mu(X))^{1/s}.\end{aligned}$$

Now let us choose r and s. We take

$$r = \frac{p}{q} > 1 \implies \frac{1}{r} = \frac{q}{p} \qquad \text{and} \qquad \frac{1}{s} = 1 - \frac{1}{r} = 1 - \frac{q}{p},$$

hence

$$\begin{aligned}\|u\|_q &= \left(\int |u|^p \, d\mu \right)^{q/p \cdot 1/q} \cdot (\mu(X))^{(1-q/p)(1/q)} \\ &= \left(\int |u|^p \, d\mu \right)^{q/p \cdot 1/q} \cdot (\mu(X))^{1/q-1/p} \\ &= \|u\|_p \cdot (\mu(X))^{1/q-1/p}.\end{aligned}$$

(ii) If $u \in \mathcal{L}^p$ we know that u is measurable and $\|u\|_p < \infty$. The inequality in (i) then shows that

$$\|u\|_q \leqslant \text{const} \cdot \|u\|_p < \infty,$$

hence $u \in \mathcal{L}^q$. This gives $\mathcal{L}^p \subset \mathcal{L}^q$. The inclusion $\mathcal{L}^q \subset \mathcal{L}^1$ follows by taking $p \rightsquigarrow q$, $q \rightsquigarrow 1$.

Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^p$ be a Cauchy sequence, i.e. $\lim_{m,n\to\infty} \|u_n - u_m\|_p = 0$. Since by the inequality in (i) also

$$\lim_{m,n\to\infty} \|u_n - u_m\|_q \leqslant \mu(X)^{1/q-1/p} \lim_{m,n\to\infty} \|u_n - u_m\|_p = 0$$

we get that $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^q$ is also a Cauchy sequence in $\mathcal{L}^q$.

(iii) No, the assertion breaks down completely if the measure μ has infinite mass. Here is an example: $\mu =$ Lebesgue measure on $(1, \infty)$. Then the function $f(x) = \frac{1}{x}$ is not integrable over $[1, \infty)$, but $f^2(x) = \frac{1}{x^2}$ is. In other words: $f \notin \mathcal{L}^1(1, \infty)$ but $f \in \mathcal{L}^2(1, \infty)$, hence $\mathcal{L}^2(1, \infty) \not\subset \mathcal{L}^1(1, \infty)$. (Playing around with different exponents shows that the assertion also fails for other $p, q \geqslant 1$....).

Problem 12.2 This is going to be a bit messy and rather than showing the 'streamlined' solution we indicate how one could find out the numbers oneself. Now let λ be some number in $(0, 1)$ and let α, β be conjugate indices: $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ where $\alpha, \beta \in (1, \infty)$. Then by the Hölder inequality

$$\begin{aligned} \int |u|^r \, d\mu &= \int |u|^{r\lambda} |u|^{r(1-\lambda)} \, d\mu \\ &\leqslant \left(\int |u|^{r\lambda\alpha} \, d\mu \right)^{\frac{1}{\alpha}} \left(\int |u|^{r(1-\lambda)\beta} \, d\mu \right)^{\frac{1}{\beta}} \\ &= \left(\int |u|^{r\lambda\alpha} \, d\mu \right)^{\frac{r\lambda}{r\lambda\alpha}} \left(\int |u|^{r(1-\lambda)\beta} \, d\mu \right)^{\frac{r(1-\lambda)}{r(1-\lambda)\beta}}. \end{aligned}$$

Taking rth roots on both sides yields

$$\begin{aligned} \|u\|_r &\leqslant \left(\int |u|^{r\lambda\alpha} \, d\mu \right)^{\frac{\lambda}{r\lambda\alpha}} \left(\int |u|^{r(1-\lambda)\beta} \, d\mu \right)^{\frac{(1-\lambda)}{r(1-\lambda)\beta}} \\ &= \|u\|_{r\lambda\alpha}^{\lambda} \|u\|_{r(1-\lambda)\beta}^{1-\lambda}. \end{aligned}$$

This leads to the following system of equations:

$$p = r\lambda\alpha q \qquad\qquad = r(1-\lambda)\beta 1 = \frac{1}{\alpha} + \frac{t}{\beta}$$

with unknown quantities α, β, λ. Solving it yields

$$\lambda = \frac{\frac{1}{r} - \frac{1}{q}}{\frac{1}{p} - \frac{1}{q}}, \quad \alpha = \frac{q-p}{q-r} \quad \beta = \frac{q-p}{r-p}.$$

Problem 12.3 $v \in \mathcal{L}^\infty(\mu)$ means that $|v(x)| \leqslant (\|v\|_\infty + \epsilon)$ for all $x \in N = N_\epsilon$ with $\mu(N) = 0$. Using in step $*$ below Theorem 10.9, we get

$$\begin{aligned} \int uv \, d\mu &\leqslant \int |u||v| \, d\mu \\ &\overset{*}{=} \int_{N^c} |u||v| \, d\mu \end{aligned}$$

$$
\begin{aligned}
&\leqslant \int_{N^c} |u|\big(\|v\|_\infty + \epsilon\big)\, d\mu \\
&= \big(\|v\|_\infty + \epsilon\big) \int_{N^c} |u|\, d\mu \\
&\leqslant \big(\|v\|_\infty + \epsilon\big) \int |u|\, d\mu
\end{aligned}
$$

and since the left-hand side does not depend on $\epsilon > 0$, we can let $\epsilon \to 0$ and find

$$\int uv\, d\mu \leqslant \left| \int uv\, d\mu \right| \leqslant \int |uv|\, d\mu \leqslant \big(\|v\|_\infty + \epsilon\big)\|u\|_1 \xrightarrow{\epsilon \to 0} \|v\|_\infty \|u\|_1.$$

Problem 12.4 Proof by induction in N.

Start $N = 2$: this is just Hölder's inequality.

Hypothesis: the generalized Hölder inequality holds for some $N \geqslant 2$.

Step $N \rightsquigarrow N + 1$:. Let $u_1, \dots, u_N, w$ be $N + 1$ functions and let $p_1, \dots, p_N, q > 1$ be such that $p_1^{-1} + p_2^{-1} + \dots + p_N^{-1} + q^{-1} = 1$. Set $p^{-1} := p_1^{-1} + p_2^{-1} + \dots + p_N^{-1}$. Then, by the ordinary Hölder inequality,

$$
\begin{aligned}
\int |u_1 \cdot u_2 \cdot \ldots \cdot u_N \cdot w|\, d\mu &\leqslant \left(\int |u_1 \cdot u_2 \cdot \ldots \cdot u_N|^p\, d\mu \right)^{1/p} \|u\|_q \\
&= \left(\int |u_1|^p \cdot |u_2|^p \cdot \ldots \cdot |u_N|^p\, d\mu \right)^{1/p} \|u\|_q
\end{aligned}
$$

Now use the induction hypothesis which allows us to apply the generalized Hölder inequality for N (!) factors $\lambda_j := p/p_j$, and thus $\sum_{j=1}^N \lambda_j^{-1} = p/p = 1$, to the first factor to get

$$
\begin{aligned}
\int |u_1 \cdot u_2 \cdot \ldots \cdot u_N \cdot w|\, d\mu &= \left(\int |u_1|^p \cdot |u_2|^p \cdot \ldots \cdot |u_N|^p\, d\mu \right)^{1/p} \|u\|_q \\
&\leqslant \|u\|_{p_1} \cdot \|u\|_{p_2} \cdot \ldots \cdot \|u\|_{p_N} \|u\|_q.
\end{aligned}
$$

Problem 12.5 Draw a picture similar to the one used in the proof of Lemma 12.1 (note that the increasing function need not be convex or concave....). Without loss of generality we can assume that $A, B > 0$ are such that $\phi(A) \geqslant B$ which is equivalent to $A \geqslant \psi(B)$ since ϕ and ψ are inverses. Thus,

$$AB = \int_0^B \psi(\eta)\, d\eta + \int_0^{\psi(B)} \phi(\xi)\, d\xi + \int_{\psi(B)}^A B\, d\xi.$$

Using the fact that ϕ increases, we get that

$$\phi(\psi(B)) = B \implies \phi(C) \geqslant B \quad \forall\, C \geqslant \psi(B)$$

and we conclude that

$$\begin{aligned}
AB &= \int_0^B \psi(\eta)\, d\eta + \int_0^{\psi(B)} \phi(\xi)\, d\xi + \int_{\psi(B)}^A B\, d\xi \\
&\leqslant \int_0^B \psi(\eta)\, d\eta + \int_0^{\psi(B)} \phi(\xi)\, d\xi + \int_{\psi(B)}^A \phi(\xi)\, d\xi \\
&= \int_0^B \psi(\eta)\, d\eta + \int_0^A \phi(\xi)\, d\xi \\
&= \Psi(B) + \Phi(A).
\end{aligned}$$

Problem 12.6 Let us show first of all that $\mathcal{L}^p\text{-}\lim_{k\to\infty} u_k = u$. This follows immediately from $\lim_{k\to\infty} \|u - u_k\|_p = 0$ since the series $\sum_{k=1}^\infty \|u - u_k\|_p$ converges.

Therefore, we can find a subsequence $(u_{k(j)})_{j\in\mathbb{N}}$ such that

$$\lim_{j\to\infty} u_{k(j)}(x) = u(x) \quad \text{almost everywhere.}$$

Now we want to show that u is the a.e. limit of the original sequence. For this we mimic the trick from the Riesz-Fischer theorem 12.7 and show that the series

$$\sum_{j=0}^\infty (u_{j+1} - u_j) = \lim_{K\to\infty} \sum_{j=0}^K (u_{j+1} - u_j) = \lim_{K\to\infty} u_K$$

(again we agree on $u_0 := 0$ for notational convenience) makes sense. So let us employ Lemma 12.6 used in the proof of the Riesz-Fischer theorem to get

$$\begin{aligned}
\left\| \sum_{j=0}^\infty (u_{j+1} - u_j) \right\|_p &\leqslant \left\| \sum_{j=0}^\infty |u_{j+1} - u_j| \right\|_p \\
&\leqslant \sum_{j=0}^\infty \|u_{j+1} - u_j\|_p \\
&\leqslant \sum_{j=0}^\infty \left(\|u_{j+1} - u\|_p + \|u - u_j\|_p \right)
\end{aligned}$$

$$< \infty$$

where we used Minkowski's inequality, the function u from above and the fact that $\sum_{j=1}^{\infty} \|u_j - u\|_p < \infty$ along with $\|u_1\|_p < \infty$. This shows that $\lim_{K\to\infty} u_K(x) = \sum_{j=0}^{\infty}(u_{j+1}(x) - u_j(x))$ exists almost everywhere.

We still have to show that $\lim_{K\to\infty} u_K(x) = u(x)$. For this we remark that a subsequence has necessarily the same limit as the original sequence—whenever both have limits, of course. But then,

$$u(x) = \lim_{j\to\infty} u_{k(j)}(x) = \lim_{k\to\infty} u_k(x) = \sum_{j=0}^{\infty}(u_{j+1}(x) - u_j(x))$$

and the claim follows.

Problem 12.7 That for every fixed x the sequence

$$u_n(x) := n\mathbf{1}_{(0,1/n)}(x) \xrightarrow{n\to\infty} 0$$

is obvious. On the other hand, for any subsequence $(u_{n(j)})_j$ we have

$$\int |u_{n(j)}|^p \, d\lambda = n(j)^p \, \frac{1}{n(j)} = n(j)^{p-1} \xrightarrow{j\to\infty} c$$

with $c = 1$ in case $p = 1$ and $c = \infty$ if $p > 1$. This shows that the $\mathcal{L}^p$-limit of this subsequence—let us call it w if it exists at all—cannot be (not even a.e.) $u = 0$.

On the other hand, we know that a sub-subsequence $(\tilde{u}_{k(j)})_j$ of $(u_{k(j)})_j$ converges pointwise almost everywhere to the $\mathcal{L}^p$-limit:

$$\lim_j \tilde{u}_{k(j)}(x) = w(x).$$

Since the full sequence $\lim_n u_n(x) = u(x) = 0$ has a limit, this shows that the sub-sub-sequence limit $w(x) = 0$ almost everywhere—a contradiction. Thus, w does not exist in the first place.

Problem 12.8 Using Minkowski's and Hölder's inequalities we find for all $\epsilon > 0$

$$\begin{aligned}
\|u_k v_k - uv\|_1 &= \|u_k v_k - u_k v + u_k v - uv\| \\
&\leqslant \|u_k \cdot (v_k - v)\| + \|(u_k - u)v\| \\
&\leqslant \|u_k\|_p \|v_k - v\|_q + \|u_k - u\|_p \|v\|_q \\
&\leqslant (M + \|v\|_q)\epsilon
\end{aligned}$$

for all $n \geqslant N_\epsilon$. We used here that the sequence $(\|u_k\|_p)_{k\in\mathbb{N}}$ is bounded. Indeed, by Minkowski's inequality

$$\|u_k\|_p = \|u_k - u\|_p + \|u\|_p \leqslant \epsilon + \|u\|_p =: M.$$

Problem 12.9 We use the simple identity

$$\begin{aligned}\|u_n - u_m\|_2^2 &= \int (u_n - u_m)^2\,d\mu \\ &= \int (u_n^2 - 2u_n u_m + u_m)\,d\mu \qquad (*) \\ &= \|u_n\|_2^2 + \|u_m\|_2^2 - 2\int u_n u_m\,d\mu.\end{aligned}$$

Case 1: $u_n \to u$ in $\mathcal{L}^2$. This means that $(u_n)_{n\in\mathbb{N}}$ is an $\mathcal{L}^2$ Cauchy sequence, i.e. that $\lim_{m,n\to\infty}\|u_n - u_m\|_2^2 = 0$. On the other hand, we get from the lower triangle inequality for norms

$$\lim_{n\to\infty} |\|u_n\|_2 - \|u\|_2| \leqslant \lim_{n\to\infty} \|u_n - u\|_2 = 0$$

so that also $\lim_{n\to\infty}\|u_n\|_2^2 = \lim_{m\to\infty}\|u_m\|_2^2 = \|u\|_2^2$. Using (*) we find

$$\begin{aligned}2\int u_n u_m\,d\mu &= \|u_n\|_2^2 + \|u_m\|_2^2 - \|u_n - u_m\|_2^2 \\ &\xrightarrow{n,m\to\infty} \|u\|_2^2 + \|u\|_2^2 - 0 \\ &= 2\|u\|_2^2.\end{aligned}$$

Case 2: Assume that $\lim_{n,m\to\infty}\int u_n u_m\,d\mu = c$ for some number $c \in \mathbb{R}$. By the very definition of this double limit, i.e.

$$\forall\,\epsilon > 0 \quad \exists\,N_\epsilon \in \mathbb{N} \quad : \quad \left|\int u_n u_m\,d\mu - c\right| < \epsilon \quad \forall n, m \geqslant N_\epsilon,$$

we see that $\lim_{n\to\infty}\int u_n u_n\,d\mu = c = \lim_{m\to\infty}\int u_m u_m\,d\mu$ hold (with the same c!). Therefore, again by (*), we get

$$\begin{aligned}\|u_n - u_m\|_2^2 &= \|u_n\|_2^2 + \|u_m\|_2^2 - 2\int u_n u_m\,d\mu \\ &\xrightarrow{n,m\to\infty} c + c - 2c \ = \ 0,\end{aligned}$$

i.e. $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{L}^2$ and has, by the completeness of this space, a limit.

Problem 12.10 Use the exponential series to conclude from the positivity of h and $u(x)$ that

$$\exp(hu) = \sum_{j=0}^{\infty} \frac{h^j u^j}{j!} \geqslant \frac{h^N}{N!} u^N.$$

Integrating this gives

$$\frac{h^N}{N!} \int u^N \, d\mu \leqslant \int \exp(hu) \, d\mu < \infty$$

and we find that $u \in \mathcal{L}^N$. Since μ is a finite measure we know from Problem 12.1 that for $N > p$ we have $\mathcal{L}^N \subset \mathcal{L}^p$.

Problem 12.11 (i) We have to show that $|u_n(x)|^p := n^{p\alpha}(x+n)^{-p\beta}$ has finite integral—measurability is clear since u_n is continuous. Since $n^{p\alpha}$ is a constant, we have only to show that $(x+n)^{-p\beta}$ is in $\mathcal{L}^1$. Set $\gamma := p\beta > 1$. Then we get from a Beppo-Levi and a domination argument

$$\begin{aligned}
\int_{(0,\infty)} (x+n)^{-\gamma} \, \lambda(dx) &\leqslant \int_{(0,\infty)} (x+1)^{-\gamma} \, \lambda(dx) \\
&\leqslant \int_{(0,1)} 1 \, \lambda(dx) + \int_{(1,\infty)} (x+1)^{-\gamma} \, \lambda(dx) \\
&\leqslant 1 + \lim_{k\to\infty} \int_{(1,k)} x^{-\gamma} \, \lambda(dx).
\end{aligned}$$

Now using that Riemann=Lebesgue on intervals where the Riemann integral exists, we get

$$\begin{aligned}
\lim_{k\to\infty} \int_{(1,k)} x^{-\gamma} \, \lambda(dx) &= \lim_{k\to\infty} \int_1^k x^{-\gamma} \, dx \\
&= \lim_{k\to\infty} \left[(1-\gamma)^{-1} x^{1-\gamma}\right]_1^k \\
&= (1-\gamma)^{-1} \lim_{k\to\infty} \left(k^{1-\gamma} - 1\right) \\
&= (\gamma - 1)^{-1} \; < \; \infty
\end{aligned}$$

which shows that the integral is finite.

(ii) We have to show that $|v_n(x)|^q := n^{q\gamma} e^{-qnx}$ is in $\mathcal{L}^1$—again measurability is inferred from continuity. Since $n^{q\gamma}$ is a constant, it is enough to show that e^{-qnx} is integrable. Set $\delta = qn$. Since

$$\lim_{x\to\infty} (\delta x)^2 e^{-\delta x} = 0 \qquad \text{and} \qquad e^{-\delta x} \leqslant 1 \quad \forall\, x \geqslant 0,$$

and since $e^{-\delta x}$ is continuous on $[0,\infty)$, we conclude that there are constants $C, C(\delta)$ such that

$$\begin{aligned} e^{-\delta x} &\leqslant \min\left\{1, \frac{C}{(\delta x)^2}\right\} \\ &\leqslant C(\delta)\min\left\{1, \frac{1}{x^2}\right\} \\ &= C(\delta)\left(\mathbf{1}_{(0,1)}(x) + \mathbf{1}_{[1,\infty)}\frac{1}{x^2}\right) \end{aligned}$$

but the latter is an integrable function on $(0,\infty)$.

Problem 12.12 Without loss of generality we may assume that $\alpha \leqslant \beta$. We distinguish between the case $x \in (0,1)$ and $x \in [1,\infty)$. If $x \leqslant 1$, then

$$\frac{1}{x^\alpha} \geqslant \frac{1}{x^\alpha + x^\beta} \geqslant \frac{1}{x^\alpha + x^\alpha} = \frac{1/2}{x^\alpha + x^\alpha} \quad \forall\, x \leqslant 1;$$

this shows that $(x^\alpha + x^\beta)^{-1}$ is in $\mathcal{L}^1((0,1), dx)$ if, and only if, $\alpha < 1$.

Similarly, if $x \geqslant 1$, then

$$\frac{1}{x^\beta} \geqslant \frac{1}{x^\alpha + x^\beta} \geqslant \frac{1}{x^\beta + x^\beta} = \frac{1/2}{x^\beta + x^\beta} \quad \forall\, x \geqslant 1$$

this shows that $(x^\alpha + x^\beta)^{-1}$ is in $\mathcal{L}^1((1,\infty), dx)$ if, and only if, $\beta > 1$.

Thus, $(x^\alpha + x^\beta)^{-1}$ is in $\mathcal{L}^1(\mathbb{R}, dx)$ if, and only if, both $\alpha < 1$ and $\beta > 1$.

Problem 12.13 If we use $X = \{1, 2, \dots, n\}$, $x(j) = x_j$, $\mu = \epsilon_1 + \cdots + \epsilon_n$ we have

$$\left(\sum_{j=1}^n |x_j|^p\right)^{1/p} = \|x\|_{L^p(\mu)}$$

and it is clear that this is a norm for $p \geqslant 1$ and, in view of Problem 12.18 it is not a norm for $p < 1$ since the triangle (Minkowski) inequality fails. (This could also be shown by a direct counterexample.

Problem 12.14 Without loss of generality we can restrict ourselves to positive functions—else we would consider positive and negative parts. Separability can obviously considered separately!

Assume that $\mathcal{L}^1_+$ is separable and choose $u \in \mathcal{L}^p_+$. Then $u^p \in \mathcal{L}^1$ and, because of separability, there is a sequence $(f_n)_n \subset \mathcal{D}_1 \subset \mathcal{L}^1$ such that

$$f_n \xrightarrow[n\to\infty]{\text{in } \mathcal{L}^1} u^p \implies u_n^p \xrightarrow[n\to\infty]{\text{in } \mathcal{L}^1} u^p$$

if we set $u_n := f_n^{1/p} \in \mathcal{L}^p$. In particular, $u_{n(k)}(x) \to u(x)$ almost everywhere for some subsequence and $\|u_{n(k)}\|_p \xrightarrow{k\to\infty} \|u\|_p$. Thus, Riesz' theorem 12.10 applies and proves that

$$\mathcal{L}^p \ni u_{n(k)} \xrightarrow[k\to\infty]{\text{in } \mathcal{L}^p} u.$$

Obviously the separating set $\mathcal{D}_p$ is essentially the same as $\mathcal{D}_1$, and we are done.

The converse is similar (note that we did not make any assumptions on $p \geqslant 1$ or $p < 1$—this is immaterial in the above argument).

Problem 12.15 We have seen in the lecture that, whenever $\lim_{n\to\infty} \|u - u_n\|_p = 0$, there is a subsequence $u_{n(k)}$ such that $\lim_{k\to\infty} u_{n(k)}(x) = u(x)$ almost everywhere. Since, by assumption, $\lim_{j\to\infty} u_j(x) = w(x)$ a.e., we have also that $\lim_{j\to\infty} u_{n(j)}(x) = w(x)$ a.e., hence $u(x) = w(x)$ almost everywhere.

Problem 12.16 We remark that $y \mapsto \log y$ is concave. Therefore, we can use Jensen's inequality for concave functions to get for the probability measure $\mu/\mu(X) = \mu(X)^{-1}\mathbf{1}_X\,\mu$

$$\begin{aligned} \int (\log u)\,\frac{d\mu}{\mu(X)} &\leqslant \log\left(\int u\,\frac{d\mu}{\mu(X)}\right) \\ &= \log\left(\frac{\int u\,d\mu}{\mu(X)}\right) \\ &= \log\left(\frac{1}{\mu(X)}\right), \end{aligned}$$

and the claim follows.

Problem 12.17 As a matter of fact,

$$\int_{(0,1)} u(s)\,ds \cdot \int_{(0,1)} \log u(t)\,dt \leqslant \int_{(0,1)} u(x)\log u(x)\,dx.$$

We begin by proving the hint. $\log x \geqslant 0 \iff x \geqslant 1$. So,

$$\begin{aligned} &\forall y \geqslant 1 \;:\; \Big(\log y \leqslant y\log y \iff 1 \leqslant y\Big) \\ \text{and}\quad &\forall y \leqslant 1 \;:\; \Big(\log y \leqslant y\log y \iff 1 \geqslant y\Big). \end{aligned}$$

Assume now that $\int_{(0,1)} u(x)\,dx = 1$. Substituting in the above inequality $y = u(x)$ and integrating over $(0,1)$ yields

$$\int_{(0,1)} \log u(x)\,dx \leqslant \int_{(0,1)} u(x)\log u(x)\,dx.$$

Now assume that $\alpha = \int_{(0,1)} u(x)\,dx$. Then $\int_{(0,1)} u(x)/\alpha\,dx = 1$ and the above inequality gives

$$\int_{(0,1)} \log\frac{u(x)}{\alpha}\,dx \leqslant \int_{(0,1)} \frac{u(x)}{\alpha}\log\frac{u(x)}{\alpha}\,dx$$

which is equivalent to

$$\begin{aligned}
&\int_{(0,1)} \log u(x)\,dx - \log\alpha \\
&\qquad = \int_{(0,1)} \log u(x)\,dx - \int_{(0,1)} \log\alpha\,dx \\
&\qquad = \int_{(0,1)} \log\frac{u(x)}{\alpha}\,dx \\
&\qquad \leqslant \int_{(0,1)} \frac{u(x)}{\alpha}\log\frac{u(x)}{\alpha}\,dx \\
&\qquad = \frac{1}{\alpha}\int_{(0,1)} u(x)\log\frac{u(x)}{\alpha}\,dx \\
&\qquad = \frac{1}{\alpha}\int_{(0,1)} u(x)\log u(x)\,dx - \frac{1}{\alpha}\int_{(0,1)} u(x)\log\alpha\,dx \\
&\qquad = \frac{1}{\alpha}\int_{(0,1)} u(x)\log u(x)\,dx - \frac{1}{\alpha}\int_{(0,1)} u(x)\,dx\log\alpha \\
&\qquad = \frac{1}{\alpha}\int_{(0,1)} u(x)\log u(x)\,dx - \log\alpha.
\end{aligned}$$

The claim now follows by adding $\log\alpha$ on both sides and then multiplying by $\alpha = \int_{(0,1)} u(x)\,dx$.

Problem 12.18 Note the misprint: $q = p/(p-1) \iff \frac{1}{p} + \frac{1}{q} = 1$ independent of $p \in (1,\infty)$ or $p \in (0,1)$!

(i) Let $p \in (0,1)$ and pick the conjugate index $q := p/(p-1) < 0$. Moreover, $s := 1/p \in (1,\infty)$ and the conjugate index t, $\frac{1}{s} + \frac{1}{t} = 1$, is given by

$$t = \frac{s}{s-1} = \frac{\frac{1}{p}}{\frac{1}{p}-1} = \frac{1}{1-p} \in (1,\infty).$$

Thus, using the normal Hölder inequality for s, t we get

$$\begin{aligned}\int u^p\,d\mu &= \int u^p\,\frac{w^p}{w^p}\,d\mu\\ &\leqslant \left(\int \left(u^p w^p\right)^s\,d\mu\right)^{1/s}\left(\int w^{-pt}\,d\mu\right)^{1/t}\\ &= \left(\int uw\,d\mu\right)^p\left(\int w^{p/(p-1)}\,d\mu\right)^{1-p}.\end{aligned}$$

Taking pth roots on either side yields

$$\begin{aligned}\left(\int u^p\,d\mu\right)^{1/p} &\leqslant \left(\int uw\,d\mu\right)\left(\int w^{p/(p-1)}\,d\mu\right)^{(1-p)/p}\\ &= \left(\int uw\,d\mu\right)\left(\int w^q\,d\mu\right)^{-1/q}\end{aligned}$$

and the claim follows.

(ii) This ‘reversed’ Minkowski inequality follows from the ‘reversed’ Hölder inequality in exactly the same way as Minkowski’s inequality follows from Hölder’s inequality, cf. Corollary 12.4. To wit:

$$\begin{aligned}\int (u+v)^p\,d\mu &= \int (u+v)\cdot(u+v)^{p-1}\,d\mu\\ &= \int u\cdot(u+v)^{p-1}\,d\mu + \int v\cdot(u+v)^{p-1}\,d\mu\\ &\overset{\text{(i)}}{\geqslant} \|u\|_p\cdot\left\|(u+v)^{p-1}\right\|_q + \|v\|_p\cdot\left\|(u+v)^{p-1}\right\|_q.\end{aligned}$$

Dividing both sides by $\||u+v|^{p-1}\|_q$ proves our claim since

$$\left\|(u+v)^{p-1}\right\|_q = \left(\int (u+v)^{(p-1)q}\,d\mu\right)^{1/q} = \left(\int (u+v)^p\,d\mu\right)^{1-1/p}.$$

Problem 12.19 By assumption, $|u| \leqslant \|u\|_\infty \leqslant C < \infty$ and $u \not\equiv 0$.

(i) We have

$$M_n = \int |u|^n\,d\mu \leqslant C^n \int d\mu = C^n \mu(X) \in (0,\infty).$$

Note that $M_n > 0$.

(ii) By the Cauchy-Schwarz-Inequality,

$$\begin{aligned}
M_n &= \int |u|^n \, d\mu \\
&= \int |u|^{\frac{n+1}{2}} |u|^{\frac{n-1}{2}} \, d\mu \\
&\leqslant \left(\int |u|^{n+1} \, d\mu \right)^{1/2} \left(\int |u|^{n-1} \, d\mu \right)^{1/2} \\
&= \sqrt{M_{n+1} M_{n-1}}\,.
\end{aligned}$$

(iii) The upper estimate follows from

$$M_{n+1} = \int |u|^{n+1} \, d\mu \leqslant \int |u|^n \cdot \|u\|_\infty \, d\mu = \|u\|_\infty M_n.$$

Set $P := \mu/\mu(X)$; the lower estimate is equivalent to

$$\begin{aligned}
&\left(\int |u|^n \, \frac{d\mu}{\mu(X)} \right)^{1/n} \leqslant \frac{\int |u|^{n+1} \, \frac{d\mu}{\mu(X)}}{\int |u|^n \, \frac{d\mu}{\mu(X)}} \\
\iff &\left(\int |u|^n \, dP \right)^{1+1/n} \leqslant \int |u|^{n+1} \, dP \\
\iff &\left(\int |u|^n \, dP \right)^{(n+1)/n} \leqslant \int |u|^{n+1} \, dP
\end{aligned}$$

and the last inequality follows easily from Jensen's inequality since P is a probability measure:

$$\left(\int |u|^n \, dP \right)^{(n+1)/n} \int |u|^{n \cdot \frac{n+1}{n}} \, dP = \int |u|^{n+1} \, dP.$$

(iv) Following the hint we get

$$\|u\|_n \geqslant \Big(\mu\{u > \|u\|_\infty - \epsilon\} \Big)^{1/n} \big(\|u\|_\infty - \epsilon \big) \xrightarrow[\epsilon \to 0]{n \to \infty} \|u\|_\infty,$$

i.e.

$$\liminf_{n \to \infty} \|u\|_n \geqslant \|u\|_\infty.$$

Combining this with the estimate from (iii), we get

$$\|u\|_\infty \leqslant \liminf_{n \to \infty} \mu(X)^{-1/n} \|u\|_n$$

$$
\begin{aligned}
&\overset{\text{(iii)}}{\leqslant} \liminf_{n\to\infty} \frac{M_{n+1}}{M_n}\\
&\leqslant \limsup_{n\to\infty} \frac{M_{n+1}}{M_n}\\
&\leqslant \|u\|_\infty.
\end{aligned}
$$

Problem 12.20 The hint says it all.... Maybe, you have a look at the specimen solution of Problem 12.19, too.

Problem 12.21 Without loss of generality we may assume that $f \geqslant 0$. We use the following standard representation of f, see (8.7):

$$f = \sum_{j=0}^{N} \phi_j \mathbf{1}_{A_j}$$

with $0 = \phi_0 < \phi_1 < \ldots < \phi_N < \infty$ and mutually disjoint sets A_j. Clearly, $\{f \neq 0\} = A_1 \uplus \cdots \uplus A_N$.

Assume first that $f \in \mathcal{E} \cap \mathcal{L}^p(\mu)$. Then

$$\infty > \int f^p \, d\mu = \sum_{j=1}^{N} \phi_j^p \, \mu(A_j) \geqslant \sum_{j=1}^{N} \phi_1^p \, \mu(A_j) = \phi_1^p \, \mu(\{f \neq 0\});$$

thus $\mu(\{f \neq 0\}) < \infty$.

Conversely, if $\mu(\{f \neq 0\}) < \infty$, we get

$$\int f^p \, d\mu = \sum_{j=1}^{N} \phi_j^p \, \mu(A_j) \leqslant \sum_{j=1}^{N} \phi_N^p \, \mu(A_j) = \phi_N^p \, \mu(\{f \neq 0\}) < \infty.$$

Since this integrability criterion does not depend on $p \geqslant 1$, it is clear that $\mathcal{E}^+ \cap \mathcal{L}^p(\mu) = \mathcal{E}^+ \cap \mathcal{L}^1(\mu)$, and the rest follows since $\mathcal{E} = \mathcal{E}^+ - \mathcal{E}^+$.

Problem 12.22 (i) Note that $\Lambda(x) = x^{1/q}$ is concave—e.g. differentiate twice and show that it is negative—and using Jensen's inequality for positive $f, g \geqslant 0$ yields

$$
\begin{aligned}
\int fg \, d\mu &= \int g f^{-p/q} \, \mathbf{1}_{\{f\neq 0\}} f^p \, d\mu\\
&\leqslant \int f^p \, d\mu \left(\frac{\int g^q f^{-p} \mathbf{1}_{\{f\neq 0\}} f^p \, d\mu}{\int f^p \, d\mu} \right)^{1/q}
\end{aligned}
$$

$$\leqslant \left(\int f^p \, d\mu\right)^{1-1/q} \left(\int g^q \, d\mu\right)^{1/q}$$

where we used $\mathbf{1}_{\{f\neq 0\}} \leqslant 1$ in the last step. Note that $fg \in \mathcal{L}^1$ follows from the fact that $\big(g^q f^{-p} \mathbf{1}_{\{f\neq 0\}}\big) f^p = g^q \in \mathcal{L}^1$.

(ii) The function $\Lambda(x) = (x^{1/p} + 1)^p$ has second derivative

$$\Lambda''(x) = \frac{1-p}{p}\left(1 + x^{-1/p}\right) x^{-1-1/p} \leqslant 0$$

showing that Λ is concave. Using Jensen's inequality gives for $f, g \geqslant 0$

$$\begin{aligned}
\int (f+g)^p \mathbf{1}_{\{f\neq 0\}} \, d\mu &= \int \left(\frac{g}{f}\, \mathbf{1}_{\{f\neq 0\}} + 1\right)^p f^p \mathbf{1}_{\{f\neq 0\}} \, d\mu \\
&\leqslant \int_{\{f\neq 0\}} f^p \, d\mu \left[\left(\frac{\int g^p \mathbf{1}_{\{f\neq 0\}} \, d\mu}{\int_{\{f\neq 0\}} f^p \, d\mu}\right)^{1/p} + 1\right]^p \\
&= \left[\left(\int_{\{f\neq 0\}} g^p \, d\mu\right)^{1/p} + \left(\int_{\{f\neq 0\}} f^p \, d\mu\right)^{1/p}\right]^p.
\end{aligned}$$

Adding on both sides $\int_{\{f=0\}} (f+g)^p \, d\mu = \int_{\{f=0\}} g^p \, d\mu$ yields, because of the elementary inequality $A^p + B^p \leqslant (A+B)^p$, $A, B \geqslant 0$, $p \geqslant 1$,

$$\begin{aligned}
&\int (f+g)^p \, d\mu \\
&\leqslant \left[\left(\int_{\{f\neq 0\}} g^p \, d\mu\right)^{1/p} + \left(\int_{\{f\neq 0\}} f^p \, d\mu\right)^{1/p}\right]^p + \left[\int_{\{f=0\}} g^p \, d\mu\right]^{p/p} \\
&\leqslant \left[\left(\int g^p \, d\mu\right)^{1/p} + \left(\int f^p \, d\mu\right)^{1/p}\right]^p.
\end{aligned}$$