13 Product measures and Fubini's theorem Solutions to Problems 13.1–13.14

Problem 13.1 • We have

$$
\begin{aligned}
(x,y) \in \Big(\bigcup_i A_i\Big) \times B &\iff x \in \bigcup_i A_i \text{ and } y \in B\\
&\iff \exists\, i_0 \;:\; x \in A_{i_0} \text{ and } y \in B\\
&\iff \exists\, i_0 \;:\; (x,y) \in A_{i_0} \times B\\
&\iff (x,y) \in \bigcup_i (A_i \times B).
\end{aligned}
$$

• We have

$$
\begin{aligned}
(x,y) \in \Big(\bigcap_i A_i\Big) \times B &\iff x \in \bigcap_i A_i \text{ and } y \in B\\
&\iff \forall\, i \;:\; x \in A_i \text{ and } y \in B\\
&\iff \forall\, i \;:\; (x,y) \in A_i \times B\\
&\iff (x,y) \in \bigcap_i (A_i \times B).
\end{aligned}
$$

• Using the formula $A \times B = \pi_1^{-1}(A) \cap \pi_2^{-1}(B)$ (see page 120 and the fact that inverse maps interchange with all set operations, we get

$$
\begin{aligned}
(A \times B) \cap (A' \times B') &= \Big[\pi_1^{-1}(A) \cap \pi_2^{-1}(B)\Big] \cap \Big[\pi_1^{-1}(A') \cap \pi_2^{-1}(B')\Big]\\
&= \Big[\pi_1^{-1}(A) \cap \pi_1^{-1}(A')\Big] \cap \Big[\pi_2^{-1}(B) \cap \pi_2^{-1}(B')\Big]\\
&= \pi_1^{-1}(A \cap A') \cap \pi_2^{-1}(B \cap B')\\
&= (A \cap A') \times (B \cap B').
\end{aligned}
$$

• Using the formula $A \times B = \pi_1^{-1}(A) \cap \pi_2^{-1}(B)$ (see page 120 and the fact that inverse maps interchange with all set operations, we get

$$
A^c \times B = \pi_1^{-1}(A^c) \cap \pi_2^{-1}(B)
$$

$$
\begin{aligned}
&= \big[\pi_1^{-1}(A)\big]^c \cap \pi_2^{-1}(B)\\
&= \pi_1^{-1}(X) \cap \pi_2^{-1}(B) \cap \big[\pi_1^{-1}(A)\big]^c\\
&= \pi_1^{-1}(X) \cap \pi_2^{-1}(B) \cap \Big\{\big[\pi_1^{-1}(A)\big]^c \cup \big[\pi_2^{-1}(B)\big]^c\Big\}\\
&= (X \times B) \cap \big[\pi_1^{-1}(A) \cap \pi_2^{-1}(B)\big]^c\\
&= (X \times B) \cap \big[A \times B\big]^c\\
&= (X \times B) \setminus (A \times B).
\end{aligned}
$$

- We have

$$
\begin{aligned}
A \times B \subset A' \times B' &\iff \big[(x,y) \in A \times B \implies (x,y) \in A' \times B'\big]\\
&\iff \big[x \in A, y \in B \implies x \in A', y \in B'\big]\\
&\iff A \subset A',\quad B \subset B'.
\end{aligned}
$$

Problem 13.2 Pick two exhausting sequences $(A_k)_k \subset \mathcal{A}$ and $(B_k)_k \subset \mathcal{B}$ such that $\mu(A_k), \nu(B_k) < \infty$ and $A_k \uparrow X$, $B_k \uparrow Y$. Then, because of the continuity of measures,

$$
\begin{aligned}
\mu \times \nu(A \times N) &= \lim_k \mu \times \nu\big((A \times N) \cap (A_k \times B_k)\big)\\
&= \lim_k \mu \times \nu\big((A \cap A_k) \times (N \cap B_k)\big)\\
&= \lim_k \big[\underbrace{\mu(A \cap A_k)}_{<\infty} \cdot \underbrace{\nu(N \cap B_k)}_{\leqslant \nu(N) = 0}\big]\\
&= 0.
\end{aligned}
$$

Since $A \times N \in \mathcal{A} \times \mathcal{B} \subset \mathcal{A} \otimes \mathcal{B}$, measurability is clear.

Problem 13.3 Since the two expressions are symmetric in x and y, they must coincide if they converge. Let us, therefore only look at the left hand side.

The inner integral,

$$\int_{(0,\infty)} e^{-xy} \sin x \, \lambda(dx)$$

clearly satisfies

$$
\begin{aligned}
\int_{(0,\infty)} \big|e^{-xy} \sin x\big| \, \lambda(dx) &\leqslant \int_{(0,\infty)} e^{-xy} \, \lambda(dx)\\
&= \int_0^\infty e^{-xy} \, dx
\end{aligned}
$$

$$= \left[-\frac{e^{-xy}}{y} \right]_{x=0}^{\infty}$$
$$= \frac{1}{x}.$$

Since the integrand is continuous and has only one sign, we can use Riemann's integral. Thus, the integral exists. To calculate its value we observe that two integrations by parts yield

$$\begin{aligned}
\int_0^\infty e^{-xy}\, sinx\, dx &= -e^{-xy}\cos x\Big|_{x=0}^{\infty} - \int_0^\infty ye^{-xy}\cos x\, dx \\
&= 1 - y\int_0^\infty e^{-xy}\cos x\, dx \\
&= 1 - y\left(e^{-xy}\sin x\Big|_{x=0}^{\infty} + \int_0^\infty ye^{-xy}\sin x\, dx \right) \\
&= 1 - y^2\int_0^\infty e^{-xy}\sin x\, dx.
\end{aligned}$$

And if we solve this equality for the integral expression, we get

$$(1+y^2)\int_0^\infty e^{-xy}\sin x\, dx = 1 \implies \int_0^\infty e^{-xy}\sin x\, dx = \frac{1}{1+y^2}.$$

Alternative: Since $\sin x = \operatorname{Im} e^{ix}$ we get

$$\int_0^\infty e^{-xy}\, sinx\, dx = \operatorname{Im}\int_0^\infty e^{-(y-i)x}\, dx = \operatorname{Im}\frac{1}{y-i} = \operatorname{Im}\frac{y+i}{y^2+1} = \frac{1}{y^2+1}.$$

Thus the iterated integral exists, since

$$\int_{(0,\infty)} \left| \frac{\sin x}{1+x^2} \right| dx \leqslant \int_{(0,\infty)} \frac{1}{1+x^2}\, dx = \arctan x\Big|_0^\infty = \frac{\pi}{2}.$$

(Here we used again that improper Riemann integrals with positive integrands coincide with Lebesgue integrals.)

In principle, the existence and equality of iterated integrals is not good enough to guarantee the existence of the double integral. For this one needs the existence of the *absolute* iterated integrals—cf. Tonelli's theorem 13.8. In the present case one can see that the absolute iterated integrals exist, though:

On the one hand we find

$$\int_{(0,\infty)} e^{-xy}|\sin(x)|\, \lambda(dx) \leqslant \frac{e^{-xy}}{-y}\bigg|_0^\infty = \frac{1}{y}$$

and $\frac{\sin y}{y}$ is, as a bounded continuous function, Lebesgue integrable over $(0,1)$.

On the other hand we can use integration by parts to get

$$\begin{aligned}\int_{k\pi}^{(k+1)\pi} e^{-xy}\sin x\,dx &= \frac{e^{-xy}}{-y}\sin x\Big|_{k\pi}^{(k+1)\pi} - \int_{k\pi}^{(k+1)\pi}\frac{e^{-xy}}{-y}\cos x\,dx\\ &= \frac{e^{-xy}}{-y^2}\cos x\Big|_{k\pi}^{(k+1)\pi} - \int_{k\pi}^{(k+1)\pi}\frac{e^{-xy}}{-y^2}(-1)\sin x\,dx\end{aligned}$$

which is equivalent to

$$\begin{aligned}\frac{y^2+1}{y^2}\int_{k\pi}^{(k+1)\pi} e^{-xy}\sin x\,dx &= \frac{e^{-(k+1)\pi y}}{-y^2}(-1)^{k+1} - \frac{e^{-k\pi y}}{-y^2}(-1)^k\\ &= \frac{(-1)^k}{y^2}(e^{-(k+1)\pi y}+e^{-k\pi y}),\end{aligned}$$

i.e. $\int_{k\pi}^{(k+1)\pi} e^{-xy}\sin x\;dx = (-1)^k\frac{1}{y^2+1}(e^{-(k+1)\pi y}+e^{-k\pi y})$.

Now we find a bound for $y\in(1,\infty)$.

$$\begin{aligned}\int_{(0,\infty)} e^{-xy}|\sin(x)|dx &= \sum_{k=0}^{\infty}\int_{k\pi}^{(k+1)\pi} e^{-xy}\sin x(-1)^k\;dx\\ &= \sum_{k=0}^{\infty}(-1)^k(-1)^k\frac{1}{y^2+1}(e^{-(k+1)\pi y}+e^{-k\pi y})\\ &\leqslant \frac{2}{y^2+1}\sum_{k=0}^{\infty}(e^{-\pi y})^k\\ &\overset{y>1}{\leqslant} \frac{2}{y^2+1}\sum_{k=0}^{\infty}(e^{-\pi})^k\end{aligned}$$

which means that the left hand side is integrable over $(1,\infty)$.

Thus we have

$$\begin{aligned}&\int_{(0,\infty)}\int_{(0,\infty)} |e^{-xy}\sin x\sin y|\,\lambda(dx)\,\lambda(dy)\\ &\qquad\leqslant \int_{(0,1]}\frac{\sin y}{y}\;\lambda(dy) + \int_{(1,\infty)}\frac{2}{y^2+1}\,\lambda(dy)\sum_{k=0}^{\infty}(e^{-\pi})^k\\ &\qquad< \infty.\end{aligned}$$

By Fubini's theorem we know that the iterated integrals as well as the double integral exist and their values are identical.

Problem 13.4 Note that

$$\frac{d}{dy}\frac{y}{x^2+y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}\,.$$

Thus we can compute

$$\int_{(0,1)}\int_{(0,1)} \frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx = \int_{(0,1)} \frac{1}{x^2+1}\,dx = \arctan x\Big|_0^1 = \frac{\pi}{4}.$$

By symmetry of x and y in the integrals it follows that

$$\int_{(0,1)}\int_{(0,1)} \frac{y^2-x^2}{(x^2+y^2)^2}\,dy\,dx = -\frac{\pi}{4}$$

and therefore the double integral can not exist. Since the existence would imply the equality of the two above integrals. We can see this directly by

$$\begin{aligned}\int_{(0,1)}\int_{(0,1)} \left|\frac{x^2-y^2}{(x^2+y^2)^2}\right|\,dy\,dx &\geqslant \int_0^1\int_0^x \frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx \\ &= \int_0^1 \frac{x}{x^2+x^2}\,dx \\ &= \frac{1}{2}\int_0^1 \frac{1}{x}\,dx = \infty.\end{aligned}$$

Problem 13.5 Since the integrand is odd, we have for $y \neq 0$:

$$\int_{(-1,1)} \frac{xy}{(x^2+y^2)^2}\,dx = 0$$

and $\{0\}$ is a null set. Thus the iterated integrals have common value 0. But the double integral does not exist, since for the iterated absolute integrals we get

$$\int_{(-1,1)} \left|\frac{xy}{(x^2+y^2)^2}\right|\,dx = \frac{1}{|y|}\int_0^{1/|y|} \frac{\xi}{(\xi^2+1)^2}\,d\xi \;\geqslant \frac{2}{|y|}\underbrace{\int_0^1 \frac{\xi}{(\xi^2+1)^2}\,d\xi}_{<\infty}.$$

Here we used the substitution $x = \xi|y|$ and the fact that $|y| \leqslant 1$, thus $1/|y| \geqslant 1$. But the outer integral is bounded below by

$$\int_{(-1,1)} \frac{2}{|y|}\,dy \quad \text{which is divergent.}$$

Problem 13.6 (i) Since for continuous integrands over a compact interval Riemann and Lebesgue integrals coincide, we find

$$\begin{aligned}\lim_{k\to\infty}\int_{(0,k)} e^{-tx}\,\lambda(dt) &= \lim_{k\to\infty}\int_{[0,k]} e^{-tx}\,dt\\ &= \lim_{k\to\infty}\frac{e^{-tx}}{-x}\bigg|_0^k\\ &= \lim_{n\to\infty}\frac{e^{-kx}}{-x}-\frac{1}{-x}=\frac{1}{x}\,.\end{aligned}$$

(ii) Since $|\sin x\int_{(0,k)} e^{-tx}\,dt| \leqslant |\frac{\sin x}{x}|$ and since $\sin x/x$ is continuous and bounded on the interval $[0,n]$, we can use dominated convergence to get

$$\begin{aligned}\lim_{n\to\infty}\int_{(0,n)}\frac{\sin x}{x}\,\lambda(dx) &= \lim_{n\to\infty}\int_{(0,n)}\sin x\,\lim_{k\to\infty}\int_{(0,k)} e^{-tx}\,dt\,dx\\ &= \lim_{n\to\infty}\lim_{k\to\infty}\int_{(0,n)}\int_{(0,k)}\sin x\,e^{-tx}\,dt\,dx.\end{aligned}$$

Since the integrand is continuous and since we integrate over a (relatively) compact set we can use Fubini's theorem and find

$$\begin{aligned}&\lim_{n\to\infty}\int_{(0,n)}\frac{\sin x}{x}\,\lambda(dx)\\ &\quad= \lim_{n\to\infty}\lim_{k\to\infty}\int_{(0,k)}\int_{(0,n)}\sin x\,e^{-tx}\,dx\,dt\\ &\quad= \lim_{n\to\infty}\lim_{k\to\infty}\int_{(0,k)}\frac{1}{t^2+1}\Big(1-e^{-nt}(\cos n+t\sin n)\Big)\,dt\end{aligned}$$

where we also used that

$$\int_a^b e^{-xy}\sin x\,dx = \frac{1}{y^2+1}\big(e^{-ay}(\cos a+y\sin a)-e^{-by}(\cos b+y\sin b)\big)$$

Since

$$\begin{aligned}&\left|\mathbf{1}_{(0,k)}(t)\frac{1}{t^2+1}\big(1-e^{-nt}(\cos n+t\sin n)\big)\right|\\ &\qquad\leqslant \frac{2}{t^2+1}+\frac{t}{t^2+1}e^{-nt}\in L^1(0,\infty),\end{aligned}$$

dominated convergence yields

$$\lim_{n\to\infty}\int_{(0,n)}\frac{\sin x}{x}\,\lambda(dx) = \lim_{n\to\infty}\int_{(0,\infty)}\frac{1}{t^2+1}\Big(1-e^{-nt}(\cos n+t\sin n)\Big)\,dt$$

and, again by dominated convergence, since the integrand is for $n > 1$ bounded by the integrable function $(0,\infty) \ni t \mapsto \frac{2}{t^2+1} + \frac{t}{t^2+1}\, e^{-t}$

$$\lim_{n\to\infty} \int_{(0,n)} \frac{\sin x}{x}\,\lambda(dx) = \int_{(0,\infty)} \frac{1}{t^2+1}\,dt = \arctan t\Big|_0^\infty = \frac{\pi}{2}\,.$$

Problem 13.7 Note that the diagonal $\Delta \subset \mathbb{R}^2$ is measurable, i.e. the (double) integrals are well-defined. The inner integral on the l.h.S. satisfies

$$\int_{[0,1]} \mathbf{1}_\Delta(x,y)\,\lambda(dx) = \lambda(\{y\}) = 0 \quad \forall\, y \in [0,1]$$

so that the left-hand side

$$\int_{[0,1]}\int_{[0,1]} \mathbf{1}_\Delta(x,y)\,\lambda(dx)\,\mu(dy) = \int_{[0,1]} 0\,\mu(dy) = 0.$$

On the other hand, the inner integral on the right-hand side equals

$$\int_{[0,1]} \mathbf{1}_\Delta(x,y)\,\mu(dy) = \mu(\{x\}) = 1 \quad \forall\, x \in [0,1]$$

so that the right-hand side

$$\int_{[0,1]}\int_{[0,1]} \mathbf{1}_\Delta(x,y)\,\mu(dy)\,\lambda(dx) = \int_{[0,1]} 1\,\lambda(dx) = 1.$$

This shows that the double integrals are not equal. This does not contradict Tonelli's theorem since μ is not σ-finite.

Problem 13.8 (i) Note that, due to the countability of $\mathbb{N}$ and $\mathbb{N}\times\mathbb{N}$ there are no problems with measurability and σ-finiteness (of the counting measure).

Tonelli's Theorem. Let $(a_{jk})_{j,k\in\mathbb{N}}$ be a double sequence of positive numbers $a_{jk} \geqslant 0$. Then

$$\sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}} a_{jk} = \sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}} a_{jk}$$

with the understanding that both sides are either finite or infinite.

Fubini's Theorem. Let $(a_{jk})_{j,k\in\mathbb{N}} \subset \mathbb{R}$ be a double sequence of real numbers a_{jk}. If

$$\sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}} |a_{jk}| \quad\text{or}\quad \sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}} |a_{jk}|$$

is finite, then all of the following expressions converge absolutely and sum to the same value:

$$\sum_{j\in\mathbb{N}}\Big(\sum_{k\in\mathbb{N}}|a_{jk}|\Big),\quad \sum_{k\in\mathbb{N}}\Big(\sum_{j\in\mathbb{N}}|a_{jk}|\Big),\quad \sum_{(j,k)\in\mathbb{N}\times\mathbb{N}}|a_{jk}|.$$

(ii) Consider the (obviously σ-finite) measures $\mu_j := \sum_{k\in A_j}\delta_k$ and $\nu = \sum_{j\in\mathbb{N}}\mu_j$. Tonelli's theorem tells us that

$$\begin{aligned}
\sum_{j\in\mathbb{N}}\sum_{k\in A_j}|x_k| &= \int_{\mathbb{N}}\int_{\mathbb{N}}|x_k|\,\mu_j(dk)\,\mu(dj)\\
&= \int_{\mathbb{N}}\int_{\mathbb{N}}|x_k|\,\mathbf{1}_{A_j}(k)\,\mu(dk)\,\mu(dj)\\
&= \int_{\mathbb{N}}\int_{\mathbb{N}}|x_k|\,\mathbf{1}_{A_j}(k)\,\mu(dj)\,\mu(dk)\\
&= \int_{\mathbb{N}}|x_k|\,\underbrace{\Big(\int_{\mathbb{N}}\mathbf{1}_{A_j}(k)\,\mu(dj)\Big)}_{=1,\ \text{as the } A_j \text{ are disjoint}}\,\mu(dk)\\
&= \int_{\mathbb{N}}|x_k|\,\mu(dk)\\
&= \sum_{k\in\mathbb{N}}|x_k|.
\end{aligned}$$

Problem 13.9 (i) Set $U(a,b) := a-b$. Then

$$U(u(x),y)\mathbf{1}_{[0,\infty)}(y)\geqslant 0 \iff u(x)\geqslant y\geqslant 0$$

and $U(u(x),y)\mathbf{1}_{[0,\infty)}(y)$ is a combination/sum/product of $\mathcal{B}(\mathbb{R}^2)$ resp. $\mathcal{B}(\mathbb{R})$-measurable functions. Thus $S[u]$ is $\mathcal{B}(\mathbb{R}^2)$-measurable.

(ii) Yes, true, since by Tonelli's theorem

$$\begin{aligned}
\lambda^2(S[u]) &= \int_{\mathbb{R}^2}\mathbf{1}_{S[u]}(x,y)\,\lambda^2(d(x,y))\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\mathbf{1}_{\{(x,y)\,:\,u(x)\geqslant y\geqslant 0\}}(x,y)\,\lambda^1(dy)\,\lambda^1(dx)\\
&= \int_{\mathbb{R}}\int_{[0,u(x)]}1\,\lambda^1(dy)\,\lambda^1(dx)\\
&= \int_{\mathbb{R}}u(x)\,\lambda^1(dx)
\end{aligned}$$

(iii) Measurability follows from (i) and with the hint. Moreover,

$$\begin{aligned}
\lambda^2(\Gamma[u]) &= \int_{\mathbb{R}^2} \mathbf{1}_{\Gamma[u]}(x,y)\,\lambda^2(d(x,y)) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{1}_{\{(x,y)\,:\,y=u(x)\}}(x,y)\,\lambda^1(dy)\,\lambda^1(dx) \\
&= \int_{\mathbb{R}}\int_{[u(x),u(x)]} 1\,\lambda^1(dy)\,\lambda^1(dx) \\
&= \int_{\mathbb{R}} \lambda^1(\{u(x)\})\,\lambda^1(dx) \\
&= \int_{\mathbb{R}} 0\,\lambda^1(dx) \\
&= 0.
\end{aligned}$$

Problem 13.10 The hint given in the text should be good enough to solve this problem....

Problem 13.11 Since (i) implies (ii), we will only prove (i) under the assumption that both $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are complete measure spaces. Note that we have to assume σ-finiteness of μ and ν, otherwise the product construction would not work. Pick some set $Z \in \mathcal{P}(X) \setminus \mathcal{A}$ (which is, because of completeness, not a null-set!), and some ν-null set $N \in \mathcal{B}$ and consider $Z \times N$.

We get for some exhausting sequence $(A_k)_k \subset \mathcal{A}$, $A_k \uparrow X$ and $\mu(A_k) < \infty$:

$$\begin{aligned}
\mu \times \nu(X \times N) &= \sup_{k\in\mathbb{N}} \mu \times \nu(A_k \times N) \\
&= \sup_{k\in\mathbb{N}} \big(\underbrace{\mu(A_k)}_{<\infty} \cdot \underbrace{\nu(N)}_{=0} \big) \\
&= 0;
\end{aligned}$$

thus $Z \times N \subset X \times N$ is a subset of a measurable $\mu \times \nu$ null set, hence it should be $\mathcal{A} \otimes \mathcal{B}$-measurable, if the product space were complete. On the other hand, because of Theorem 13.10(iii), if $Z \times N$ is $\mathcal{A} \otimes \mathcal{B}$-measurable, then the section

$$x \mapsto \mathbf{1}_{Z\times N}(x,y) = \mathbf{1}_Z(x)\mathbf{1}_N(y) \overset{y\in N}{=} \mathbf{1}_Z(x)$$

is $\mathcal{A}$-measurable which is only possible if $Z \in \mathcal{A}$.

Problem 13.12 (i) Let $A \in \mathcal{B}[0,\infty) \otimes \mathcal{P}(\mathbb{N})$, fix $k \in \mathbb{N}$ and consider

$$\mathbf{1}_A(x,k) \quad \text{and} \quad B_k := \{x \,:\, \mathbf{1}_A(x,k) = 1\};$$

because of Theorem 13.10(iii), $B_k \in \mathcal{B}[0,\infty)$. Since

$$\begin{aligned}(x,k) \in A &\iff \mathbf{1}_A(x,k) = 1\\ &\iff \exists\, k \in \mathbb{N} \,:\, \mathbf{1}_A(x,k) = 1\\ &\iff \exists\, k \in \mathbb{N} \,:\, x \in B_k\end{aligned}$$

it is clear that $A = \bigcup_{k\in\mathbb{N}} B_k \times \{k\}$.

(ii) Let $M \in \mathcal{P}(\mathbb{N})$ and set $\zeta := \sum_{j\in\mathbb{N}} \delta_j$; we know that ζ is a (σ-finite) measure on $\mathcal{P}(\mathbb{N})$. Using Tonelli's theorem 13.8 we get

$$\begin{aligned}\pi(B \times M) &:= \sum_{m\in M} \pi(B \times \{m\})\\ &:= \sum_{m\in M} \int_B e^{-t}\,\frac{t^m}{m!}\,\mu(dt)\\ &= \int_M \int_B e^{-t}\,\frac{t^m}{m!}\,\mu(dt)\,\zeta(dm)\\ &= \iint_{B\times M} e^{-t}\,\frac{t^m}{m!}\,\mu\times\zeta(dt,dm)\end{aligned}$$

which shows that the measure $\pi(dt,dm) := e^{-t}\,\frac{t^m}{m!}\,\mu\times\zeta(dt,dm)$ has all the properties required by the exercise.

The uniqueness follows, however, from the uniqueness theorem for measures (Theorem 5.7): the family of 'rectangles' of the form $B \times M \in \mathcal{B}[0,\infty) \times \mathcal{P}(\mathbb{N})$ is a $\cap$-stable generator of the product σ-algebra $\mathcal{B}[0,\infty) \otimes \mathcal{P}(\mathbb{N})$ and contains an exhausting sequence, say, $[0,\infty) \times \{1,2,\dots k\} \uparrow [0,\infty) \times \mathbb{N}$. But on this generator π is (uniquely) determined by prescribing the values $\pi(B \times \{m\})$.

Problem 13.13 (i) This is similar to Problem 7.9, in particular (i) and (vi).

(ii) Note that

$$\begin{aligned}\mathbf{1}_B(x,y) &= \mathbf{1}_{(a,b]}(x)\mathbf{1}_{[x,b]}(y)\\ &= \mathbf{1}_{(a,b]}(y)\mathbf{1}_{(a,y]}(x)\\ &= \mathbf{1}_{(a,b]}(x)\mathbf{1}_{(a,b]}(y)\mathbf{1}_{[0,\infty)}(y-x);\end{aligned}$$

the last expression is, however, a product of (combinations of) measurable functions, thus $\mathbf{1}_B$ is measurable and so is then B.

Without loss of generality we can assume that $a > 0$, all other cases are similar.

Using Tonelli's theorem 13.8 we get

$$\begin{aligned}
\mu \times \nu(B) &= \iint \mathbf{1}_B(x,y)\,\mu \times \nu(dx,dy) \\
&= \iint \mathbf{1}_{(a,b]}(y)\mathbf{1}_{(a,y]}(x)\,\mu \times \nu(dx,dy) \\
&= \int_{(a,b]} \int_{(a,y]} \mu(dx)\,\nu(dy) \\
&= \int_{(a,b]} \mu(a,y]\,\nu(dy) \\
&= \int_{(a,b]} \big(\mu(0,y] - \mu(0,a]\big)\,\nu(dy) \\
&= \int_{(a,b]} \mu(0,y]\nu(dy) - \mu(0,a]\int_{(a,b]} \nu(dy) \\
&= \int_{(a,b]} F(y)\,dG(y) - F(a)\big(G(b) - G(a)\big). \qquad (*)
\end{aligned}$$

We remark at this point already that a very similar calculation (with μ, ν and F, G interchanged and with an open interval rather than a semi-open interval) yields

$$\begin{aligned}
&\iint \mathbf{1}_{(a,b]}(y)\mathbf{1}_{(y,b]}(x)\,\nu(dy)\,\mu(dx) \\
&\qquad = \int_{(a,b]} G(y-)\,dF(y) - G(a)\big(F(b) - F(a)\big).
\end{aligned} \qquad (**)$$

(iii) On the one hand we have

$$\begin{aligned}
\mu \times \nu\big((a,b] \times (a,b]\big) &= \mu(a,b]\nu(a,b] \\
&= \big(F(b) - F(a)\big)\big(G(b) - G(a)\big)
\end{aligned} \qquad (+)$$

and on the other we find, using Tonelli's theorem at step (T)

$$\begin{aligned}
&\mu \times \nu\big((a,b] \times (a,b]\big) \\
&\quad = \iint \mathbf{1}_{(a,b]}(x)\mathbf{1}_{(a,b]}(y)\,\mu(dx)\,\nu(dy)
\end{aligned}$$

$$
\begin{aligned}
&= \iint \mathbf{1}_{(a,b]}(x)\mathbf{1}_{(x,b]}(y)\,\mu(dx)\,\nu(dy)+\\
&\qquad + \iint \mathbf{1}_{(a,b]}(x)\mathbf{1}_{(a,x)}(y)\,\mu(dx)\,\nu(dy)\\
&\overset{T}{=} \iint \mathbf{1}_{(a,b]}(x)\mathbf{1}_{(x,b]}(y)\,\mu(dx)\,\nu(dy)+\\
&\qquad + \iint \mathbf{1}_{(a,b]}(y)\mathbf{1}_{(y,b]}(x)\,\nu(dy)\,\mu(dx)\\
&\overset{*,**}{=} \int_{(a,b]} F(y)\,dG(y) - F(a)\big(G(b)-G(a)\big)+\\
&\qquad + \int_{(a,b]} G(y-)\,dF(y) - G(a)\big(F(b)-F(a)\big).
\end{aligned}
$$

Combining this formula with the previous one marked (+) reveals that

$$
F(b)G(b) - F(a)G(a) = \int_{(a,b]} F(y)\,dG(y) + \int_{(a,b]} G(y-)\,dF(y).
$$

Finally, observe that

$$
\begin{aligned}
\int_{(a,b]} \big(F(y)-F(y-)\big)\,dG(y) &= \int_{(a,b]} \mu(\{y\})\,\nu(dy)\\
&= \sum_{a<y\leqslant b} \mu(\{y\})\nu(\{y\})\\
&= \sum_{a<y\leqslant b} \Delta F(y)\Delta G(y).
\end{aligned}
$$

(Mind that the sum is at most countable because of Lemma 13.12) from which the claim follows.

(iv) It is clear that uniform approximation allows to interchange limiting and integration procedures so that we *really* do not have to care about this. We show the formula for monomials $t, t^2, t^3, \ldots$ by induction. Write $\phi_n(t) = t^n$, $n \in \mathbb{N}$.

Induction start $n = 1$: in this case $\phi_1(t) = t$, $\phi_1'(t) = 1$ and $\phi(F(s)) - \phi(F(s-)) - \Delta F(s) = 0$, i.e. the formula just becomes

$$
F(b) - F(a) = \int_{(a,b]} dF(s)
$$

which is obviously true.

Induction assumption: for some n we know that

$$\begin{aligned}\phi_n(F(b)) - \phi_n(F(a)) &= \int_{(a,b]} \phi_n'(F(s-))\,dF(s)\\ &\quad + \sum_{a<s\leqslant b}\Big[\phi_n(F(s)) - \phi_n(F(s-)) - \phi_n'(F(s-))\Delta F(s)\Big].\end{aligned}$$

Mind the misprint in the statement of the problem!

Induction step $n \rightsquigarrow n+1$: Write, for brevity $F = F(s)$ and $F_- = F(s-)$. We have because of (iii) with $G = \phi_n \circ F$ and because of the induction assumption

$$\begin{aligned}&\phi_{n+1}(F(b)) - \phi_{n+1}(F(a))\\ &= F(b)\phi_n(F(b)) - F(a)\phi_n(F(a))\\ &= \int_{(a,b]} F_-^n\,dF + \int_{(a,b]} F_-\,dF^n + \sum \Delta F \Delta F^n\\ &= \int_{(a,b]} F_-^n\,dF + \int_{(a,b]} F_-\,\phi_n'(F_-)\,dF +\\ &\qquad + \sum\Big[F_-\phi_n(F) - F_-\phi_n(F_-) - F_-\phi_n'(F_-)\Delta F\Big] + \sum \Delta F\Delta F^n\\ &= \int_{(a,b]} F_-^n\,dF + \int_{(a,b]} F_-\,nF_-^{n-1}\,dF +\\ &\qquad + \sum\Big[F_-F^n - F_-^{n+1} - F_-nF_-^{n-1}\Delta F + \Delta F\Delta F^n\Big]\\ &= \int_{(a,b]} (n+1)F_-^n\,dF + \sum\Big[F_-F^n - F_-^{n+1} - nF_-^n\Delta F + \Delta F\Delta F^n\Big]\\ &= \int_{(a,b]} \phi_{n+1}' \circ F_-\,dF + \sum\Big[F_-F^n - F_-^{n+1} - nF_-^n\Delta F + \Delta F\Delta F^n\Big]\end{aligned}$$

The expression under the sum can be written as

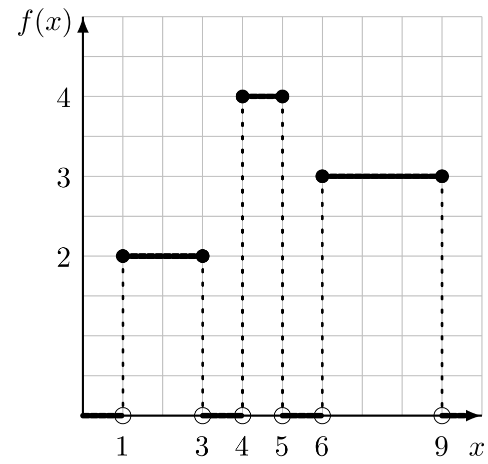
$$\begin{aligned}&F_-F^n - F_-^{n+1} - nF_-^n\Delta F + \Delta F\Delta F^n\\ &\quad = (F_- - F)F^n + F^{n+1} - F_-^{n+1} - nF_-^n\Delta F + \Delta F\Delta F^n\\ &\quad = F^{n+1} - F_-^{n+1} + \Delta F\Big(-F^n - nF_-^n + \Delta F^n\Big)\\ &\quad = F^{n+1} - F_-^{n+1} + \Delta F\Big(-F^n - nF_-^n + F^n - F_-^n\Big)\\ &\quad = F^{n+1} - F_-^{n+1} - (n+1)F_-^n\Delta F\\ &\quad = \phi_{n+1}\circ F - \phi_{n+1}\circ F_- - \phi_{n+1}'\circ F_-\Delta F\end{aligned}$$

and the induction is complete.

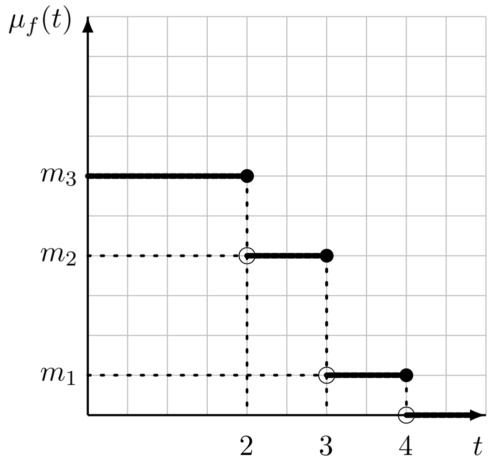
Problem 13.14 Mind the misprint in the problem:

$$\mu_f(t) := \mu(\{|f| \geqslant t\}).$$

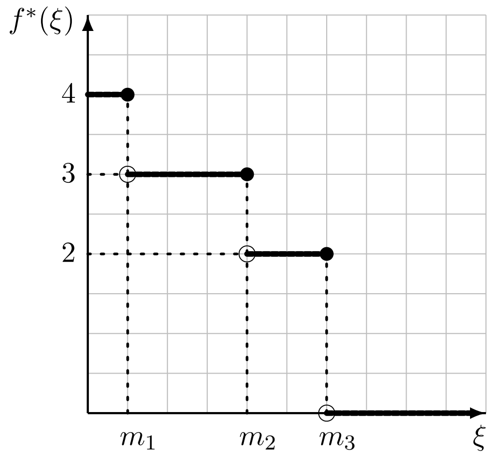
(i) We find the following pictures:



This is the graph of the original function $f(x)$. Open and full dots indicate the continuity behaviour at the jump points. x-values are to be measured in μ-length, i.e. x is a point in the measure space $(X, \mathcal{A}, \mu)$.



This is the graph of the associated distribution function $\mu_f(t)$. It is decreasing and left-continuous at the jump points. t-values are to be measured using Lebesgue measure in $[0,\infty)$.
$m_1 = \mu\big([4,5]\big)$
$m_2 - m_1 = \mu\big([6,9]\big)$
$m_3 - m_2 = \mu\big([4,5]\big)$



This is the graph of the decreasing rearrangement $f^*(\xi)$ of $f(x)$. It is decreasing and left-continuous at the jump points. ξ-values are to be measured using Lebesgue measure in $[0,\infty)$. m_1, m_2, m_3 are as in the previous picture.

(ii) The first equality,

$$\int_{\mathbb{R}} |f|^p \, d\mu = p \int_0^\infty t^{p-1} \, \mu_f(t) \, dt,$$

follows immediately from Theorem 13.11 with $u = |f|$ and $\mu_f(t) = \mu(\{|f| \geqslant t\})$.
To show the second equality we have two possibilities. We can...

a) ...show the second equality first for (positive) elementary functions and use then a (by now standard...) Beppo Levi/monotone convergence argument to extend the result to all positive measurable functions. Assume that $f(x) = \sum_{j=0}^N a_j \mathbf{1}_{B_j}(x)$ is a positive elementary function in standard representation, i.e. $a_0 = 0 < a_1 < \cdots < a_n < \infty$ and the sets $B_j = \{f = a_j\}$ are pairwise disjoint. Then we have

$$\begin{aligned}
\mu(\{f = a_j\}) &= \mu(\{f \geqslant a_j\} \setminus \{f \geqslant a_{j+1}\}) \\
&= \mu(\{f \geqslant a_j\}) - \mu(\{f \geqslant a_{j+1}\}) \\
&= \mu_f(a_j) - \mu_f(a_{j+1}) \qquad (a_{n+1} := \infty, \mu_f(a_{n+1}) = 0) \\
&= \lambda^1\big((\mu_f(a_{j+1}), \mu_f(a_j)]\big) \\
&= \lambda^1(f^* = a_j).
\end{aligned}$$

This proves

$$\int f^p \, d\mu = \sum_{j=0}^n a_j^p \, \mu(B_j) = \sum_{j=0}^n a_j^p \, \lambda^1(f^* = a_j) = \int (f^*)^p \, d\lambda^1$$

and the general case follows from the above-mentioned Beppo Levi argument.

or we can

b) use Theorem 13.11 once again with $u = f^*$ and $\mu = \lambda^1$ provided we know that

$$\mu\big(\{|f| \geqslant t\}\big) = \lambda^1\big(\{f^* \geqslant t\}\big).$$

This, however, follows from

$$f^*(\xi) \geqslant t \iff \inf\{s \,:\, \mu_f(s) \leqslant \xi\} \geqslant t$$

$$\begin{aligned} &\iff \mu_f(t) \leqslant \xi \qquad \text{(as } \mu_f \text{ is left cts. \& decreasing)} \\ &\iff \mu\big(\{|f| \geqslant t\}\big) \leqslant \xi \end{aligned}$$

and therefore

$$\lambda^1\big(\{\xi \,:\, f^*(\xi) \geqslant t\}\big) = \lambda^1\big(\{\xi \,:\, \mu(|f| \geqslant t) \leqslant \xi\}\big) = \mu(|f| \geqslant t).$$

14 Integrals with respect to image measures Solutions to Problems 14.1–14.11

Problem 14.1 The first equality

$$\int u\,d(T(f\,\mu)) = \int u \circ T\,f\,d\mu$$

is just Theorem 14.1 combined with Lemma 10.8 the formula for measures with a density.

The second equality

$$\int u \circ T\,f\,d\mu = \int u\,f \circ T^{-1}\,dT(\mu)$$

is again Theorem 14.1.

The third equality finally follows again from Lemma 10.8.

Problem 14.2 We have for any $C \in \mathcal{B}$

$$\begin{aligned}
T(\mu)|_B(C) &= T(\mu)(B \cap C) \\
&= \mu\big(T^{-1}(B \cap C)\big) \\
&= \mu\big(T^{-1}(B) \cap T^{-1}(C)\big) \\
&= \mu\big(A \cap T^{-1}(C)\big) \\
&= \mu|_A\big(T^{-1}(C)\big) \\
&= T(\mu|_A)(C).
\end{aligned}$$

Problem 14.3 By definition, we find for any Borel set $B \in \mathcal{B}(\mathbb{R}^n)$

$$\begin{aligned}
\delta_x \star \delta_y(B) &= \iint \mathbf{1}_B(s+t)\,\delta_x(ds)\,\delta_y(dt) \\
&= \int \mathbf{1}_B(x+t)\,\delta_y(dt) \\
&= \mathbf{1}_B(x+y) \\
&= \int {}_B(z)\,\delta_{x+y}(dz)
\end{aligned}$$

which means that $\delta_x \star \delta_y = \delta_{x+y}$. Note that, by Tonelli's theorem the order of the iterated integrals is irrelevant.

Similarly, since $z + t \in B \iff t \in B - z$, we find

$$\begin{aligned}\delta_z \star \mu(B) &= \iint \mathbf{1}_B(s+t)\,\delta_z(ds)\,\mu(dt)\\ &= \int \mathbf{1}_B(z+t)\,\mu(dt)\\ &= \int \mathbf{1}_{B-z}(t)\,\mu(dt)\\ &= \mu(B-z)\\ &= \tau_{-z}(\mu)(B)\end{aligned}$$

where $\tau_z(t) := \tau(t-z)$ is the shift operator so that $\tau_{-z}^{-1}(B) = B - z$.

Problem 14.4 Since $x + y \in B \iff x \in B - y$, we can rewrite formula (14.9) in the following way:

$$\begin{aligned}\mu \star \nu(B) &= \iint \mathbf{1}_B(x+y)\,\mu(dx)\,\nu(dy)\\ &= \int \left[\int \mathbf{1}_{B-y}(x)\,\mu(dx)\right]\nu(dy)\\ &= \int \mu(B-y)\,\nu(dy).\end{aligned}$$

Similarly we get

$$\mu \star \nu(B) = \int \mu(B-y)\,\nu(dy) = \int \nu(B-x)\,\mu(dx).$$

Thus, if μ has no atoms, i.e. if $\mu(\{z\}) = 0$ for all $z \in \mathbb{R}^n$, we find

$$\mu \star \nu(\{z\}) = \int \mu\big(\{z\} - y\big)\,\nu(dy) = \int \underbrace{\mu\big(\{z-y\}\big)}_{=0}\,\nu(dy) = 0.$$

Problem 14.5 Because of Tonelli's theorem we can iterate the very definition of 'convolution' of two measures, Definition 14.4(iii), and get

$$\mu_1 \star \cdots \star \mu_n(B) = \int \cdots \int \mathbf{1}_B(x_1 + \cdots + x_n)\,\mu_1(dx_1)\cdots\mu_n(dx_n)$$

so that the formula derived at the end of Remark 14.5(ii), page 138, applies and yields

$$\int |\omega|\,P^{\star n}(d\omega)$$

$$\begin{aligned}
&= \int \cdots \int |\omega_1 + \omega_2 + \cdots + \omega_n| \, P(d\omega_1) \, P(d\omega_2) \cdots P(d\omega_n) \\
&\overset{*}{\leqslant} \int \cdots \int \Big(|\omega_1| + |\omega_2| + \cdots + |\omega_n| \Big) \, P(d\omega_1) \, P(d\omega_2) \cdots P(d\omega_n) \\
&= \sum_{j=1}^{n} \int \cdots \int |\omega_j| \, P(d\omega_1) \, P(d\omega_2) \cdots P(d\omega_n) \\
&= \sum_{j=1}^{n} \int |\omega_j| \, P(d\omega_j) \cdot \prod_{k \neq j} \int P(d\omega_k) \\
&= \sum_{j=1}^{n} \int |\omega_j| \, P(d\omega_j) \\
&= n \int |\omega_1| \, P(d\omega_1)
\end{aligned}$$

where we used the symmetry of the iterated integrals in the integrating measures as well as the fact that $P(\mathbb{R}^n) = \int P(d\omega_k) = 1$. Note that we can have $+\infty$ on either side.

The equality $\int \omega \, P^{\star n}(d\omega) = n \int \omega \, P(d\omega)$ follows with same calculation (note that we do not get an inequality as there is no need for the triangle inequality at point (*) above). The integrability condition is now needed since the integrands are no longer positive. Note that, since $\omega \in \mathbb{R}^n$, the above equality is an equality between vectors in $\mathbb{R}^n$; this is no problem, just read the equality coordinate-by-coordinate.

Problem 14.6 Since the convolution $p \mapsto u \star p$ is linear, it is enough to consider monomials of the form $p(x) = x^k$. Thus, by the binomial formula,

$$\begin{aligned}
u \star p(x) &= \int u(x-y) \, y^k \, dy \\
&= \int u(y) \, (x-y)^k \, dy \\
&= \sum_{j=0}^{k} \binom{k}{j} x^j \int u(y) \, y^{k-j} \, dy.
\end{aligned}$$

Since $\operatorname{supp} u$ is compact, there is some $r > 0$ such that $\operatorname{supp} u \subset B_r(0)$ and we get for any $m \in \mathbb{N}_0$, and in particular for $m = k - j$ or $m = k$, that

$$\left| \int u(y) y^m \, dy \right| \leqslant \int_{\operatorname{supp} u} \|u\|_\infty |y|^m \, dy$$

$$\begin{aligned}&\leqslant \int_{B_r(0)} \|u\|_\infty r^m \, dy\\ &= 2r \cdot r^m \cdot \|u\|_\infty\end{aligned}$$

which is clearly finite. This shows that $u \star p$ exists and that it is a polynomial.

Problem 14.7 That the convolution $u \star w$ is bounded and continuous follows from Theorem 14.8.

Monotonicity follows from the monotonicity of the integral: if $x \leqslant z$, then

$$u \star w(x) = \int \underbrace{u(y)}_{\geqslant 0} \cdot \underbrace{w(x-y)}_{\leqslant w(z-y)} \, dy \leqslant \int u(y) \cdot w(z-y) \, dy = u \star w(y).$$

Problem 14.8 (This solution is written for $u \in C_c(\mathbb{R}^n)$ and $w \in C^\infty(\mathbb{R}^n)$).

Let $\partial_j = \partial/\partial x_j$ denote the partial derivative in direction x_j where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Since

$$w \in C^\infty \implies \partial_j w \in C^\infty,$$

it is enough to show $\partial_j(u \star w) = u \star \partial_j w$ and to iterate this equality. In particular, we find $\partial^\alpha(u \star w) = u \star \partial^\alpha w$ where

$$\partial^\alpha = \frac{\partial^{\alpha_1 + \cdots \alpha_n}}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n}, \quad \alpha \in \mathbb{N}_0^n.$$

Since u has compact support and since the derivative is a local operation (i.e., we need to know a function only in a neighbourhood of the point where we differentiate), and since we have for any $r > 0$

$$\sup_{y \in \operatorname{supp} u} \sup_{x \in B_r(0)} \left| \tfrac{\partial}{\partial x_j} w(x-y) \right| \leqslant c(r),$$

we can use the differentiability lemma for parameter-dependent integrals, Theorem 11.5 to find for any $x \in B_{r/2}(0)$, say,

$$\begin{aligned}\frac{\partial}{\partial x_j} \int u(y) w(x-y) \, dy &= \int u(y) \frac{\partial}{\partial x_j} w(x-y) \, dy\\ &= \int u(y) \big(\tfrac{\partial}{\partial x_j} w\big)(x-y) \, dy\\ &= u \star \partial_j w(x).\end{aligned}$$

Problem 14.9 The measurability considerations are just the same as in Theorem 14.6, so we skip this part.

By assumption,

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}\,;$$

We can rewrite this as

$$\frac{1}{r} + \underbrace{\left[\frac{1}{p} - \frac{1}{r}\right]}_{=1-\frac{1}{q}\in[0,1)} + \underbrace{\left[\frac{1}{q} - \frac{1}{r}\right]}_{=1-\frac{1}{p}\in[0,1)} = 1. \qquad (*)$$

Now write the integrand appearing in the definition of $u \star w(x)$ in the form

$$|u(x-y)w(y)| = \left[|u(x-y)|^{p/r}|w(y)|^{q/r}\right] \cdot \left[|u(x-y)|^{1-p/r}\right] \cdot \left[|w(y)|^{1-q/r}\right]$$

and apply the generalized Hölder inequality (cf. Problem 12.4) with the exponents from (*):

$$\begin{aligned} |u \star w(x)| &\leqslant \int |u(x-y)w(y)|\,dy \\ &\leqslant \left[\int |u(x-y)|^p |w(y)|^q\,dy\right]^{\frac{1}{r}} \left[\int |u(x-y)|^p\,dy\right]^{\frac{1}{p}-\frac{1}{r}} \left[\int |w(y)|^p\,dy\right]^{\frac{1}{q}-\frac{1}{r}}. \end{aligned}$$

Raising this inequality to the rth power we get, because of the translation invariance of Lebesgue measure,

$$\begin{aligned} |u \star w(x)|^r &\leqslant \left[\int |u(x-y)|^p |w(y)|^q\,dy\right] \|u\|_p^{r-p} \cdot \|w\|_q^{r-q} \\ &= |u|^p \star |w|^q(x) \cdot \|u\|_p^{r-p} \cdot \|w\|_q^{r-q}. \end{aligned}$$

Now we integrate this inequality over x and use Theorem 14.6 for $p = 1$ and the integral

$$\int |u|^p \star |w|^q(x)\,dx = \||u|^p \star |w|^q\|_1 \leqslant \|u\|_p^p \cdot \|w\|_q^q.$$

Thus,

$$\|u \star w\|_r^r = \int |u \star w(x)|^r\,dx \leqslant \|u\|_p^p \cdot \|w\|_q^q \cdot \|u\|_p^{r-p} \cdot \|w\|_q^{r-q} = \|u\|_p^r \cdot \|w\|_q^r$$

and the claim follows.

Problem 14.10 (i) Since ϕ is rotationally invariant, it is enough to show that the function

$$\psi(r) := e^{1/(r^2-1)}\mathbf{1}_{[-1,1]}(r)$$

is of class C^∞. This is a standard argument and we only sketch it here. Clearly, the critical points are $r = \pm 1$. Since $\psi(\pm 1) = 0 = e^{-1/0} = e^{-\infty} = 0$, the function ψ is continuous. Differentiability is shown using induction:

$$\psi'(r) = \frac{2r}{1-r^2}\, e^{1/(r^2-1)}\mathbf{1}_{[-1,1]}(r)$$

and if $\psi^{(k)}(r) = f_k(r)e^{1/(r^2-1)}\mathbf{1}_{[-1,1]}(r)$, then

$$\psi^{(k+1)}(r) = f_k'(r)e^{1/(r^2-1)}\mathbf{1}_{[-1,1]}(r) + f_k(r)\frac{2r}{1-r^2}\, e^{1/(r^2-1)}\mathbf{1}_{[-1,1]}(r).$$

This shows that $f_k(r)$ is for all $k \in \mathbb{N}$ a rational function whose growth to $\pm\infty$ as $r \to \pm 1$ is not so strong as the decay of $e^{1/(r^2-1)}$ to 0 as $r \to \pm 1$. This proves that $\psi(r)$ is arbitrarily often differentiable at the points $r = \pm 1$ (with zero derivative). For all $r \neq \pm 1$ the situation is clear.

The constant κ^{-1} is necessarily the integral of the function ϕ:

$$\kappa^{-1} = \int_{B_1(0)} \exp\left[\frac{1}{|x|^2 - 1}\right] dx\,.$$

(ii) That ϕ_ϵ is a C^∞-function is clear, since ϕ_ϵ is constructed from ϕ by a dilation.

Clearly,

$$\phi_\epsilon(x) = 0 \iff \phi(x/\epsilon) = 0 \iff |x/\epsilon| \geqslant 1 \iff |x| \geqslant \epsilon.$$

This means that $\operatorname{supp}\phi_\epsilon = \overline{B_\epsilon(0)}$.

Using Theorem 14.1 for the dilation $T = T_{1/\epsilon} : x \mapsto x/\epsilon$ and, cf. Problem 5.8 or Theorem 7.10, the fact that for Borel sets $B \in \mathcal{B}(\mathbb{R}^n)$

$$\begin{aligned}
T_{1/\epsilon}(\lambda^n)(B) &\overset{\text{def}}{=} \lambda^n(T_{1/\epsilon}^{-1}(B)) \\
&= \lambda^n(T_\epsilon(B)) \\
&= \lambda^n(\epsilon \cdot B)
\end{aligned}$$

$$\stackrel{7.10}{=} \epsilon^n \cdot \lambda^n(B),$$

we get

$$\begin{aligned}\int \phi_\epsilon(x)\,\lambda^n(dx) &= \epsilon^{-n}\int \phi(T_{1/\epsilon}(x))\,\lambda^n(dx)\\ &= \epsilon^{-n}\int \phi(x)\,T_{1/\epsilon}(\lambda^n)(dx)\\ &= \epsilon^{-n}\int \phi(x)\,\epsilon^n\cdot\lambda^n(dx)\\ &= \int \phi(x)\,\lambda^n(dx).\end{aligned}$$

(iii) We show, more generally, that

$$\operatorname{supp} u\star w \subset \operatorname{supp} u + \operatorname{supp} w \tag{*}$$

whenever $u\star w$ makes sense. Now

$$\int u(x-y)w(y)\,dy = \int_{\operatorname{supp} w} u(x-y)w(y)\,dy$$

so that

$$x-y\notin \operatorname{supp} u \iff x\notin y+\operatorname{supp} u\,.$$

Thus,

$$x\notin \operatorname{supp} u+\operatorname{supp} w \implies u\star w(x)=0.$$

Since $\operatorname{supp} u+\operatorname{supp} w$ is a closed set, we have shown (*).

(iv) The estimate

$$\|\phi_\epsilon\star u\|_p \leqslant \|\phi_\epsilon\|_1\cdot\|u\|_p \tag{**}$$

follows from Theorem 14.6.

Since $\phi_\epsilon\in C_c^\infty \implies \partial^\alpha\phi_\epsilon\in C_c^\infty$ for any $\alpha\in\mathbb{N}_0^n$. This means that $u\star\partial^\alpha\phi_\epsilon$ is well defined. However, if $p\neq 1$, we cannot appeal naively to the differentiability lemma, Theorem 11.5 to swap integration (i.e. convolution) and differentiation. To do this we consider the sequence

$$u_k(x) := \big((-k)\vee u(x)\wedge k\big)\mathbf{1}_{B_k(0)}(x)$$

and note that, by dominated convergence, L^p-$\lim_k u_k = u$ while $u_k\in L^1\cap L^\infty$. In this setting we can apply Theorem 11.5 and get

$$\partial^\alpha\big(\phi_\epsilon\star u_k(x)\big) = \frac{\partial^{\alpha_1+\cdots+\alpha_n}}{\partial^{\alpha_1}x_1\cdots\partial^{\alpha_n}x_n}\int \phi_\epsilon(x-y)u_k(y)\,dy$$

$$= \int \frac{\partial^{\alpha_1+\cdots+\alpha_n}}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n} \phi_\epsilon(x-y) u_k(y)\,dy$$
$$= \big(\partial^\alpha \phi_\epsilon\big) \star u_k(x).$$

(Note that ϕ_ϵ and u_k have compact support and are bounded functions, so domination is no problem at all.) Using the estimate (**) we find that

$$\begin{aligned}\|\partial^\alpha\big(\phi_\epsilon \star (u_k - u_\ell)\big)\|_p &= \|\big(\partial^\alpha\phi_\epsilon\big) \star (u_k - u_\ell)\|_p \\ &\leqslant \|\partial^\alpha \phi_\epsilon\|_1 \cdot \|u_k - u_\ell\|_p \xrightarrow{k,\ell\to\infty} 0.\end{aligned}$$

Since, similarly,

$$\|\big(\partial^\alpha\phi_\epsilon\big) \star (u_k - u)\|_p \leqslant \|\partial^\alpha \phi_\epsilon\|_1 \cdot \|u_k - u\|_p \xrightarrow{k\to\infty} 0,$$

we conclude that

$$\big(\partial^\alpha\phi_\epsilon\big) \star u_k \xrightarrow[L^p]{k\to\infty} \big(\partial^\alpha\phi_\epsilon\big) \star u$$

and

$$\partial^\alpha\big(\phi_\epsilon \star u_k\big) \xrightarrow[L^p]{k\to\infty} \partial^\alpha\big(\phi_\epsilon \star u\big)$$

so that

$$\partial^\alpha\big(\phi_\epsilon \star u\big) = \big(\partial^\alpha\phi_\epsilon\big) \star u.$$

(v) Since $\int \phi_\epsilon(y)\,dy = 1$, we get from Minkowski's inequality for integrals, Theorem 13.14,

$$\begin{aligned}&\|u - u \star \phi_\epsilon\|_p \\ &= \left(\int \left| \int \big(u(x) - u(x-y)\big)\phi_\epsilon(y)\,dy\right|^p dx\right)^{1/p} \\ &\leqslant \int \|u(\cdot) - u(\cdot - y)\|_p\, \phi_\epsilon(y)\,dy \\ &= \left\{ \int_{|y|\leqslant h} + \int_{|y|>h} \right\} \|u(\cdot) - u(\cdot - y)\|_p\, \phi_\epsilon(y)\,dy.\end{aligned}$$

Since the integrand $y \mapsto \|u(\cdot) - u(\cdot - y)\|_p$ is continuous, cf. Theorem 14.8, we can, for a given $\delta > 0$, pick $h = h(\delta)$ in such a way that

$$\|u(\cdot) - u(\cdot - y)\|_p \leqslant \delta \qquad \forall\, |y| \leqslant h.$$

Thus, using this estimate for the first integral term, and the triangle inequality in L^p and the translation invariance of Lebesgue integrals for the second integral expression, we get

$$\begin{aligned}\|u - u \star \phi_\epsilon\|_p &\leqslant \int\limits_{|y|\leqslant h} \delta\, \phi_\epsilon(y)\, dy + \int\limits_{|y|>h} 2\|u\|_p\, \phi_\epsilon(y)\, dy \\ &\leqslant \delta \int \phi_\epsilon(y)\, dy + 2\|u\|_p \int\limits_{|y|>h} \phi_\epsilon(y)\, dy \\ &\leqslant \delta + 2\|u\|_p \int\limits_{|y|>h} \phi_\epsilon(y)\, dy.\end{aligned}$$

Since $\operatorname{supp}\phi_\epsilon = \overline{B_\epsilon(0)}$, we can let $\epsilon \to 0$, and then $\delta \to 0$, and get

$$\limsup_{\epsilon\to 0} \|u - u \star \phi_\epsilon\|_p \leqslant \delta \xrightarrow{\delta\to 0} 0,$$

and the claim follows.

Problem 14.11 Note that $v(x) = \frac{d}{dx}(1-\cos x)\mathbf{1}_{[0,2\pi)}(x) = \mathbf{1}_{(0,2\pi)}(x)\sin x$. Thus,

(i)

$$u \star v(x) = \int_0^{2\pi} 1_{\mathbb{R}}(x-y)\sin y\, dy = \int_0^{2\pi} \sin y\, dy = 0 \quad \forall\, x.$$

(ii) Since all functions u, v, w, ϕ are continuous, we can use the usual rules for the (Riemann) integral and get, using integration by parts and the fundamental theorem of integral calculus,

$$\begin{aligned}v \star w(x) &= \int \tfrac{d}{dx}\phi(x-y) \int_{-\infty}^{x} \phi(t)\, dt\, dx \\ &= \int \big(-\tfrac{d}{dy}\phi(x-y)\big) \int_{-\infty}^{y} \phi(t)\, dt\, dx \\ &= \int \phi(x-y)\, \tfrac{d}{dy} \int_{-\infty}^{y} \phi(t)\, dt\, dx \\ &= \int \phi(x-y)\, \phi(y)\, dy \\ &= \phi \star \phi(x).\end{aligned}$$

If $x \in (0, 4\pi)$, then $x - y \in (0, 2\pi)$ for some suitable $y = y_=$ and even for all y from an interval $(y_0 - \epsilon, y_0 + \epsilon) \subset (0, 2\pi)$. Since ϕ is positive with support $[0, 2\pi]$, the positivity follows.

(iii) Obviously,

$$(u \star v) \star w \overset{(i)}{=} 0 \star w = 0$$

while

$$\begin{aligned} u \star (v \star w)(x) &= \int 1_{\mathbb{R}}(x-y) v \star w(y)\, dy \\ &= \int v \star w(y)\, dy \\ &= \int \phi \star \phi(y)\, dy \\ &> 0. \end{aligned}$$

Note that w is not an (pth power, $p < \infty$) integrable function so that we cannot use Fubini's theorem to prove associativity of the convolution.