3 σ-Algebras.
Solutions to Problems 3.1–3.12

Problem 3.1 (i) It is clearly enough to show that $A, B \in \mathcal{A} \implies A \cap B \in \mathcal{A}$, because the case of N sets follows from this by induction, the induction step being

$$\underbrace{A_1 \cap \ldots \cap A_N}_{=:B \in \mathcal{A}} \cap A_{N+1} = B \cap A_{N+1} \in \mathcal{A}.$$

Let $A, B \in \mathcal{A}$. Then, by (Σ_2) also $A^c, B^c \in \mathcal{A}$ and, by (Σ_3) and (Σ_2)

$$A \cap B = (A^c \cup B^c)^c = (A^c \cup B^c \cup \emptyset \cup \emptyset \cup \ldots)^c \in \mathcal{A}.$$

Alternative: Of course, the last argument also goes through for N sets:

$$\begin{aligned} A_1 \cap A_2 \cap \ldots \cap A_N &= (A_1^c \cup A_2^c \cup \ldots \cup A_N^c)^c \\ &= (A_1^c \cup \ldots \cup A_N^c \cup \emptyset \cup \emptyset \cup \ldots)^c \in \mathcal{A}. \end{aligned}$$

(ii) By (Σ_2) we have $A \in \mathcal{A} \implies A^c \in \mathcal{A}$. Use A^c instead of A and observe that $(A^c)^c = A$ to see the claim.

(iii) Clearly $A^c, B^c \in \mathcal{A}$ and so, by part (i), $A \setminus B = A \cap B^c \in \mathcal{A}$ as well as $A \triangle B = (A \setminus B) \cup (B \setminus A) \in \mathcal{A}$.

Problem 3.2 (iv) Let us assume that $B \neq \emptyset$ and $B \neq X$. Then $B^c \notin \{\emptyset, B, X\}$. Since with B also B^c must be contained in a σ-algebra, the family $\{\emptyset, B, X\}$ cannot be one.

(vi) Set $\mathcal{A}_E := \{E \cap A \ : \ A \in \mathcal{A}\}$. The key observation is that all set operations in $\mathcal{A}_E$ are now relative to E and not to X. This concerns mainly the complementation of sets! Let us check (Σ_1)–(Σ_3).

Clearly $\emptyset = E \cap \emptyset \in \mathcal{A}_E$. If $B \in \mathcal{A}$, then $B = E \cap A$ for some $A \in \mathcal{A}$ and the complement of B relative to E is $E \setminus B = E \cap B^c = E \cap (E \cap A)^c = E \cap (E^c \cup A^c) = E \cap A^c \in \mathcal{A}_E$ as $A^c \in \mathcal{A}$. Finally, let $(B_j)_{j\in\mathbb{N}} \subset \mathcal{A}_E$. Then there are $(A_j)_{j\in\mathbb{N}} \subset \mathcal{A}$ such that $B_j = E \cap A_j$. Since $A = \bigcup_{j\in\mathbb{N}} A_j \in \mathcal{A}$ we get $\bigcup_{j\in\mathbb{N}} B_j = \bigcup_{j\in\mathbb{N}}(E \cap A_j) = E \cap \bigcup_{j\in\mathbb{N}} A_j = E \cap A \in \mathcal{A}_E$.

(vii) Note that f^{-1} interchanges with all set operations. Let $A, A_j, j \in \mathbb{N}$ be sets in $\mathcal{A}$. We know that then $A = f^{-1}(A'), A_j = f^{-1}(A'_j)$ for suitable $A, A'_j \in \mathcal{A}'$. Since $\mathcal{A}'$ is, by assumption a σ-algebra, we have

$$\begin{aligned}
\emptyset &= f^{-1}(\emptyset) \in \mathcal{A} && \text{as} \quad \emptyset \in \mathcal{A}' \\
A^c &= \left(f^{-1}(A')\right)^c = f^{-1}(A'^c) \in \mathcal{A} && \text{as} \quad A'^c \in \mathcal{A}' \\
\bigcup_{j\in\mathbb{N}} A_j &= \bigcup_{j\in\mathbb{N}} f^{-1}(A'_j) = f^{-1}\left(\bigcup_{j\in\mathbb{N}} A'_j\right) \in \mathcal{A} && \text{as} \quad \bigcup_{j\in\mathbb{N}} A'_j \in \mathcal{A}'
\end{aligned}$$

which proves (Σ_1)–(Σ_3) for $\mathcal{A}$.

Problem 3.3 (i) Since $\mathcal{G}$ is a σ-algebra, $\mathcal{G}$ 'competes' in the intersection of all σ-algebras $\mathcal{C} \supset \mathcal{G}$ appearing in the definition of $\mathcal{A}$ in the proof of Theorem 3.4(ii). Thus, $\mathcal{G} \supset \sigma(\mathcal{G})$ while $\mathcal{G} \subset \sigma(\mathcal{G})$ is always true.

(ii) Without loss of generality we can assume that $\emptyset \neq A \neq X$ since this would simplify the problem. Clearly $\{\emptyset, A, A^c, X\}$ is a σ-algebra containing A and no element can be removed without losing this property. Thus $\{\emptyset, A, A^c, X\}$ is minimal and, therefore, $= \sigma(\{A\})$.

(iii) Assume that $\mathcal{F} \subset \mathcal{G}$. Then we have $\mathcal{F} \subset \mathcal{G} \subset \sigma(\mathcal{G})$. Now $\mathcal{C} := \sigma(\mathcal{G})$ is a potential 'competitor' in the intersection appearing in the proof of Theorem 3.4(ii), and as such $\mathcal{C} \supset \sigma(\mathcal{F})$, i.e. $\sigma(\mathcal{G}) \supset \sigma(\mathcal{F})$.

Problem 3.4 (i) $\{\emptyset, (0, \frac{1}{2}), \{0\} \cup [\frac{1}{2}, 1], [0, 1]\}$. We have 2 *atoms* (see the explanations below): $(0, \frac{1}{2}), (0, \frac{1}{2})^c$.

(ii) $\{\emptyset, [0, \frac{1}{4}), [\frac{1}{4}, \frac{3}{4}], (\frac{3}{4}, 1], [0, \frac{3}{4}], [\frac{1}{4}, 1], [0, \frac{1}{4}) \cup (\frac{3}{4}, 1], [0, 1]\}$. We have 3 *atoms* (see below): $[0, \frac{1}{4}), [\frac{1}{4}, \frac{3}{4}], (\frac{3}{4}, 1]$.

(iii) —same solution as (ii)—

Parts (ii) and (iii) are quite tedious to do and they illustrate how difficult it can be to find a σ-algebra containing two distinct sets.... imagine how to deal with something that is generated by 10, 20, or infinitely many sets. Instead of giving a particular answer, let us describe the method to find $\sigma(\{A, B\})$ practically, and then we are going to prove it.

1. Start with trivial sets and given sets: $\emptyset, X, A, B$.
2. now add their complements: A^c, B^c

3. now add their unions and intersections and differences: $A \cup B, A \cap B, A \setminus B, B \setminus A$
4. now add the complements of the sets in 3.: $A^c \cap B^c, A^c \cup B^c, (A \setminus B)^c, (B \setminus A)^c$
5. finally, add unions of differences and their complements: $(A \setminus B) \cup (B \setminus A), (A \setminus B)^c \cap (B \setminus A)^c$.

All in all one should have 16 sets (some of them could be empty or X or appear several times, depending on how much A differs from B). That's it, but the trouble is: is this construction correct? Here is a somewhat more systematic procedure:

Definition: An *atom* of a σ-algebra $\mathcal{A}$ is a non-void set $\emptyset \neq A \in \mathcal{A}$ that contains no other set of $\mathcal{A}$.

Since $\mathcal{A}$ is stable under intersections, it is also clear that all atoms are disjoint sets! Now we can make up every set from $\mathcal{A}$ as union (finite or countable) of such atoms. The task at hand is to find atoms if A, B are given. This is easy: the atoms of our future σ-algebra must be: $A \setminus B$, $B \setminus A$, $A \cap B$, $(A \cup B)^c$. (Test it: if you make a picture, this is a tesselation of our space X using disjoint sets and we can get back A, B as union! It is also minimal, since these sets must appear in $\sigma(\{A, B\})$ anyway.) The crucial point is now:

Theorem. *If $\mathcal{A}$ is a σ-algebra with N atoms (finitely many!), then $\mathcal{A}$ consists of exactly 2^N elements.*

Proof. The question is how many different unions we can make out of N sets. Simple answer: we find $\binom{N}{j}$, $0 \leqslant j \leqslant N$ different unions involving exactly j sets ($j = 0$ will, of course, produce the empty set) and they are all different as the atoms were disjoint. Thus, we get $\sum_{j=0}^{N} \binom{N}{j} = (1+1)^N = 2^N$ different sets.

It is clear that they constitute a σ-algebra. ∎

This answers the above question. The number of atoms depends obviously on the relative position of A, B: do they intersect, are they disjoint etc. Have fun with the exercises and do not try to find σ-algebras generated by three or more sets..... (By the way: can you think of a situation in $[0, 1]$ with two subsets given and exactly *four* atoms? Can there be more?)

Problem 3.5 (i) See the solution to Problem 3.4.

(ii) If $A_1, \dots, A_N \subset X$ are given, there are at most 2^N atoms. This can be seen by induction. If $N = 1$, then there are $\#\{A, A^c\} = 2$ atoms. If we add a further set A_{N+1}, then the worst case would be that A_{N+1} intersects with each of the 2^N atoms, thus splitting each atom into two sets which amounts to saying that there are $2 \cdot 2^N = 2^{N+1}$ atoms.

Problem 3.6 $\mathcal{O}_1$ Since $\emptyset$ contains no element, every element $x \in \emptyset$ admits certainly some neighbourhood $B_\delta(x)$ and so $\emptyset \in \mathcal{O}$. Since for all $x \in \mathbb{R}^n$ also $B_\delta(x) \subset \mathbb{R}^n$, $\mathbb{R}^n$ is clearly open.

$\mathcal{O}_2$ Let $U, V \in \mathcal{O}$. If $U \cap V = \emptyset$, we are done. Else, we find some $x \in U \cap V$. Since U, V are open, we find some $\delta_1, \delta_2 > 0$ such that $B_{\delta_1}(x) \subset U$ and $B_{\delta_2}(x) \subset V$. But then we can take $h := \min\{\delta_1, \delta_2\} > 0$ and find

$$B_h(x) \subset B_{\delta_1}(x) \cap B_{\delta_2}(x) \subset U \cap V,$$

i.e. $U \cap V \in \mathcal{O}$. For finitely many, say N, sets, the same argument works. Notice that already for countably many sets we will get a problem as the radius $h := \min\{\delta_j : j \in \mathbb{N}\}$ is not necessarily any longer > 0.

$\mathcal{O}_2$ Let I be any (finite, countable, not countable) index set and $(U_i)_{i \in I} \subset \mathcal{O}$ be a family of open sets. Set $U := \bigcup_{i \in I} U_i$. For $x \in U$ we find some $j \in I$ with $x \in U_j$, and since U_j was open, we find some $\delta_j > 0$ such that $B_{\delta_j}(x) \subset U_j$. But then, trivially, $B_{\delta_j}(x) \subset U_j \subset \bigcup_{i \in I} U_i = U$ proving that U is open.

The family $\mathcal{O}^n$ cannot be a σ-algebra since the complement of an open set $U \neq \emptyset, \neq \mathbb{R}^n$ is closed.

Problem 3.7 Let $X = \mathbb{R}$ and set $U_k := (-\frac{1}{k}, \frac{1}{k})$ which is an open set. Then $\bigcap_{k \in \mathbb{N}} U_k = \{0\}$ but a singleton like $\{0\}$ is closed and not open.

Problem 3.8 We know already that the Borel sets $\mathcal{B} = \mathcal{B}(\mathbb{R})$ are generated by any of the following systems:

$$\{[a,b) : a,b \in \mathbb{Q}\}, \quad \{[a,b) : a,b \in \mathbb{R}\},$$
$$\{(a,b) : a,b \in \mathbb{Q}\}, \quad \{(a,b) : a,b \in \mathbb{R}\}, \quad \mathcal{O}^1, \text{ or } \mathcal{C}^1$$

Here is just an example how to solve the problem. Let $b > a$. Since $(-\infty, b) \setminus (-\infty, a) = [a, b)$ we get that

$$\{[a,b) : a,b \in \mathbb{Q}\} \subset \sigma(\{(-\infty, c) : c \in \mathbb{Q}\})$$

$$\implies \mathcal{B} = \sigma(\{[a,b) \,:\, a,b \in \mathbb{Q}\}) \subset \sigma(\{(-\infty,c) \,:\, c \in \mathbb{Q}\}).$$

On the other hand we find that $(-\infty,a) = \bigcup_{k\in\mathbb{N}}[-k,a)$ proving that

$$\{(-\infty,a) \,:\, a \in \mathbb{Q}\} \subset \sigma(\{[c,d) \,:\, c,d \in \mathbb{Q}\}) = \mathcal{B}$$
$$\implies \sigma(\{(-\infty,a) \,:\, a \in \mathbb{Q}\}) \subset \mathcal{B}$$

and we get equality.

The other cases are similar.

Problem 3.9 Let $\mathbb{B} := \{B_r(x) \,:\, x \in \mathbb{R}^n,\ r > 0\}$ and let $\mathbb{B}' := \{B_r(x) \,:\, x \in \mathbb{Q}^n,\ r \in \mathbb{Q}^+\}$. Clearly,

$$\mathbb{B}' \subset \mathbb{B} \subset \mathcal{O}^n$$
$$\implies \sigma(\mathbb{B}') \subset \sigma(\mathbb{B}) \subset \sigma(\mathcal{O}^n) = \mathcal{B}(\mathbb{R}^n).$$

On the other hand, any open set $U \in \mathcal{O}^n$ can be represented by

$$U = \bigcup_{B\in\mathbb{B}',\, B\subset U} B. \qquad (*)$$

Indeed, $U \supset \bigcup_{B\in\mathbb{B}',\, B\subset U} B$ follows by the very definition of the union. Conversely, if $x \in U$ we use the fact that U is open, i.e. there is some $B_\epsilon(x) \subset U$. Without loss of generality we can assume that ϵ is rational, otherwise we replace it by some smaller rational ϵ. Since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$ we can find some $q \in \mathbb{Q}^n$ with $|x-q| < \epsilon/3$ and it is clear that $B_{\epsilon/3}(q) \subset B_\epsilon(x) \subset U$. This shows that $U \subset \bigcup_{B\in\mathbb{B}',\, B\subset U} B$.

Since $\#\mathbb{B}' = \#(\mathbb{Q}^n \times \mathbb{Q}) = \#\mathbb{N}$, formula $(*)$ entails that

$$\mathcal{O}^n \subset \sigma(\mathbb{B}') \implies \sigma(\mathcal{O}^n) = \sigma(\mathbb{B})$$

and we are done.

Problem 3.10 (i) $\mathcal{O}_1$: We have $\emptyset = \emptyset \cap A \in \mathcal{O}_A$, $A = X \cap A \in \mathcal{O}_A$.

$\mathcal{O}_1$: Let $U' = U \cap A \in \mathcal{O}_A$, $V' = V \cap A \in \mathcal{O}_A$ with $U, V \in \mathcal{O}$. Then $U' \cap V' = (U \cap V) \cap A \in \mathcal{O}_A$ since $U \cap V \in \mathcal{O}$.

$\mathcal{O}_2$: Let $U_i' = U_i \cap A \in \mathcal{O}_A$ with $U_i \in \mathcal{O}$. Then $\bigcup_i U_i' = (\bigcup_i U_i) \cap A \in \mathcal{O}_A$ since $\bigcup_i U_i \in \mathcal{O}$.

(ii) We use for a set A and a family $\mathcal{F} \subset \mathcal{P}(X)$ the shorthand $A \cap \mathcal{F} := \{A \cap F \,:\, F \in \mathcal{F}\}$.

Clearly, $A \cap \mathcal{O} \subset A \cap \sigma(\mathcal{O}) = A \cap \mathcal{B}(X)$. Since the latter is a σ-algebra, we have

$$\sigma(A \cap \mathcal{O}) \subset A \cap \mathcal{B}(X) \text{ i.e. } \mathcal{B}(A) \subset A \cap \mathcal{B}(X).$$

For the converse inclusion we define the family

$$\Sigma := \{B \subset X \;:\; A \cap B \in \sigma(A \cap \mathcal{O})\}.$$

It is not hard to see that Σ is a σ-algebra and that $\mathcal{O} \subset \Sigma$. Thus $\mathcal{B}(X) = \sigma(\mathcal{O}) \subset \Sigma$ which means that

$$A \cap \mathcal{B}(X) \subset \sigma(A \cap \mathcal{O}).$$

Notice that this argument does not really need that $A \in \mathcal{B}(X)$. If, however, $A \in \mathcal{B}(X)$ we have in addition to $A \cap \mathcal{B}(X) = \mathcal{B}(A)$ that

$$\mathcal{B}(A) = \{B \subset A \;:\; B \in \mathcal{B}(X)\}$$

Problem 3.11 (i) As in the proof of Theorem 3.4 we set

$$\mathfrak{m}(\mathcal{E}) := \bigcap_{\substack{\mathcal{M} \text{ monotone class} \\ \mathcal{M} \supset \mathcal{E}}} \mathcal{M}. \tag{$*$}$$

Since the intersection $\mathcal{M} = \bigcap_{i \in I} \mathcal{M}_i$ of arbitrarily many monotone classes $\mathcal{M}_i$, $i \in I$, is again a monotone class [indeed: if $(A_j)_{j\in\mathbb{N}} \subset \mathcal{M}$, then $(A_j)_{j\in\mathbb{N}}$ is in every $\mathcal{M}_i$ and so are $\bigcup_j A_j$, $\bigcap_j A_j$; thus $\bigcup_j A_j, \bigcap_j A_j \in \mathcal{M}$] and $(*)$ is evidently the smallest monotone class containing some given family $\mathcal{E}$.

(ii) Since $\mathcal{E}$ is stable under complementation and contains the empty set we know that $X \in \mathcal{E}$. Thus, $\emptyset \in \Sigma$ and, by the very definition, Σ is stable under taking complements of its elements. If $(S_j)_{j\in\mathbb{N}} \subset \Sigma$, then $(S_j^c)_{j\in\mathbb{N}} \subset \sigma$ and

$$\bigcup_j S_j \in \mathfrak{m}(\mathcal{E}), \quad \Big(\bigcup_j S_j\Big)^c = \bigcap_j S_j^c \in \mathfrak{m}(\mathcal{E})$$

which means that $\bigcup_j S_j \in \Sigma$.

(iii) $\mathcal{E} \subset \Sigma$: if $E \in \mathcal{E}$, then $E \in \mathfrak{m}(\mathcal{E})$. Moreover, as $\mathcal{E}$ is stable under complementation, $E^c \in \mathfrak{m}(\mathcal{E})$ for all $E \in \mathcal{E}$, i.e. $\mathcal{E} \subset \Sigma$.

$\Sigma \subset \mathfrak{m}(\mathcal{E})$: obvious from the definition of Σ.

$\mathfrak{m}(\mathcal{E}) \subset \sigma(\mathcal{E})$: every σ-algebra is also a monotone class and the inclusion follows from the minimality of $\mathfrak{m}(\mathcal{E})$.

Finally apply the σ-hull to the chain $\mathcal{E} \subset \Sigma \subset \mathfrak{m}(\mathcal{E}) \subset \sigma(\mathcal{E})$ and conclude that $m(\mathcal{E}) \subset \sigma(\mathcal{E})$.

Problem 3.12 (i) Since $\mathcal{M}$ is a monotone class, this follows from Problem 3.11.

(ii) Let $F \subset \mathbb{R}^n$ be any closed set. Then $U_n := F + B_{1/n}(0) := \{x+y : x \in F,\ y \in B_{1/n}(0)\}$ is an open set and $\bigcap_{n\in\mathbb{N}} U_n = F$. Indeed,

$$U_n = \bigcup_{x\in F} B_{1/n}(x) = \left\{z \in \mathbb{R}^n \ :\ |x - z| < \tfrac{1}{n} \ \text{ for some } \ x \in F\right\}$$

which shows that U_n is open, $F \subset U_n$ and $F \subset \bigcap_n U_n$. On the other hand, if $z \in U_n$ for all $n \in \mathbb{N}$, then there is a sequence of points $x_n \in F$ with the property $|z - x_n| < \frac{1}{n} \xrightarrow{n\to\infty} 0$. Since F is closed, $z = \lim_n x_n \in F$ and we get $F = \bigcap_n U_n$.

Since $\mathcal{M}$ is closed under countable intersections, $F \in \mathcal{M}$ for any closed set F.

(iii) Identical to Problem 3.11(ii).

(iv) Use Problem 3.11(iv).

4 Measures. Solutions to Problems 4.1–4.15

Problem 4.1 (i) We have to show that for a measure μ and finitely many, pairwise disjoint sets $A_1, A_2, \dots, A_N \in \mathcal{A}$ we have

$$\mu(A_1 \uplus A_2 \uplus \dots \uplus A_N) = \mu(A_1) + \mu(A_2) + \dots + \mu(A_N).$$

We use induction in $N \in \mathbb{N}$. The hypothesis is clear, for the start ($N = 2$) see Proposition 4.3(i). Induction step: take $N+1$ disjoint sets $A_1, \dots, A_{N+1} \in \mathcal{A}$, set $B := A_1 \uplus \dots \uplus A_N \in \mathcal{A}$ and use the induction start and the hypothesis to conclude

$$\begin{aligned}\mu(A_1 \uplus \dots \uplus A_N \uplus A_{N+1}) &= \mu(B \uplus A_{N+1})\\ &= \mu(B) + \mu(A_{N+1})\\ &= \mu(A_1) + \dots + \mu(A_N) + \mu(A_{N+1}).\end{aligned}$$

(iv) To get an idea what is going on we consider first the case of three sets A, B, C. Applying the formula for strong additivity thrice we get

$$\begin{aligned}\mu(A \cup B \cup C) &= \mu(A \cup (B \cup C))\\ &= \mu(A) + \mu(B \cup C) - \mu(\underbrace{A \cap (B \cup C)}_{=(A\cap B)\cup(A\cap C)})\\ &= \mu(A) + \mu(B) + \mu(C) - \mu(B \cap C) - \mu(A \cap B)\\ &\quad - \mu(A \cap C) + \mu(A \cap B \cap C).\end{aligned}$$

As an educated guess it seems reasonable to suggest that

$$\mu(A_1 \cup \dots \cup A_n) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{\sigma \subset \{1,\dots,n\}\\ \#\sigma = k}} \mu\Big(\bigcap_{j\in\sigma} A_j\Big).$$

We prove this formula by induction. The induction start is just the formula from Proposition 4.3(iv), the hypothesis is given above. For the induction step we observe that

$$\begin{aligned}\sum_{\substack{\sigma\subset\{1,\dots,n+1\}\\ \#\sigma=k}} &= \sum_{\substack{\sigma\subset\{1,\dots,n,n+1\}\\ \#\sigma=k,\ n+1\notin\sigma}} + \sum_{\substack{\sigma\subset\{1,\dots,n,n+1\}\\ \#\sigma=k,\ n+1\in\sigma}}\\ &= \sum_{\substack{\sigma\subset\{1,\dots,n\}\\ \#\sigma=k}} + \sum_{\substack{\sigma'\subset\{1,\dots,n\}\\ \#\sigma'=k-1,\ \sigma:=\sigma'\cup\{n+1\}}}\end{aligned} \qquad (*)$$

Having this in mind we get for $B := A_1 \cup \ldots \cup A_n$ and A_{n+1} using strong additivity and the induction hypothesis (for $A_1, \ldots, A_n$ resp. $A_1 \cap A_{n+1}, \ldots, A_n \cap A_{n+1}$)

$$\begin{aligned}\mu(B \cup A_{n+1}) &= \mu(B) + \mu(A_{n+1}) - \mu(B \cap A_{n+1}) \\ &= \mu(B) + \mu(A_{n+1}) - \mu\big(\bigcup_{j=1}^{n}(A_j \cap A_{n+1})\big) \\ &= \sum_{k=1}^{n}(-1)^{k+1} \sum_{\substack{\sigma\subset\{1,\ldots,n\}\\ \#\sigma=k}} \mu\big(\bigcap_{j\in\sigma} A_j\big) + \mu(A_{n+1}) \\ &\qquad + \sum_{k=1}^{n}(-1)^{k+1} \sum_{\substack{\sigma\subset\{1,\ldots,n\}\\ \#\sigma=k}} \mu\big(A_{n+1} \bigcap_{j\in\sigma} A_j\big).\end{aligned}$$

Because of $(*)$ the last line coincides with

$$\sum_{k=1}^{n+1}(-1)^{k+1} \sum_{\substack{\sigma\subset\{1,\ldots,n,n+1\}\\ \#\sigma=k}} \mu\big(\bigcap_{j\in\sigma} A_j\big)$$

and the induction is complete.

(v) We have to show that for a measure μ and finitely many sets $B_1, B_2, \ldots, B_N \in \mathcal{A}$ we have

$$\mu(B_1 \cup B_2 \cup \ldots \cup B_N) \leqslant \mu(B_1) + \mu(B_2) + \ldots + \mu(B_N).$$

We use induction in $N \in \mathbb{N}$. The hypothesis is clear, for the start ($N = 2$) see Proposition 4.3(v). Induction step: take $N + 1$ sets $B_1, \ldots, B_{N+1} \in \mathcal{A}$, set $C := B_1 \cup \ldots \cup B_N \in \mathcal{A}$ and use the induction start and the hypothesis to conclude

$$\begin{aligned}\mu(B_1 \cup \ldots \cup B_N \cup B_{N+1}) &= \mu(C \cup B_{N+1}) \\ &\leqslant \mu(C) + \mu(B_{N+1}) \\ &\leqslant \mu(B_1) + \ldots + \mu(B_N) + \mu(B_{N+1}).\end{aligned}$$

Problem 4.2 (i) The Dirac measure is defined on an arbitrary measurable space $(X, \mathcal{A})$ by $\delta_x(A) := \begin{cases} 0, & \text{if } x \notin A \\ 1, & \text{if } x \in A \end{cases}$, where $A \in \mathcal{A}$ and $x \in X$ is a fixed point.

<u>(M_1)</u> Since $\emptyset$ contains no points, $x \notin \emptyset$ and so $\delta_x(\emptyset) = 0$.

$\underline{(M_2)}$ Let $(A_j)_{j\in\mathbb{N}} \subset \mathcal{A}$ a sequence of pairwise disjoint measurable sets. If $x \in \bigcup_{j\in\mathbb{N}} A_j$, there is exactly one j_0 with $x \in A_{j_0}$, hence

$$\begin{aligned}\delta_x\left(\bigcup_{j\in\mathbb{N}} A_j\right) &= 1 = 1 + 0 + 0 + \dots \\ &= \delta_x(A_{j_0}) + \sum_{j\neq j_0} \delta_x(A_j) \\ &= \sum_{j\in\mathbb{N}} \delta_x(A_j).\end{aligned}$$

If $x \notin \bigcup_{j\in\mathbb{N}} A_j$, then $x \notin A_j$ for every $j \in \mathbb{N}$, hence

$$\delta_x\left(\bigcup_{j\in\mathbb{N}} A_j\right) = 0 = 0 + 0 + 0 + \dots = \sum_{j\in\mathbb{N}} \delta_x(A_j).$$

(ii) The measure γ is defined on $(\mathbb{R}, \mathcal{A})$ by $\gamma(A) := \begin{cases} 0, & \text{if } \#A \leqslant \#\mathbb{N} \\ 1, & \text{if } \#A^c \leqslant \#\mathbb{N} \end{cases}$

where $\mathcal{A} := \{A \subset \mathbb{R} \;:\; \#A \leqslant \#\mathbb{N} \;\text{ or }\; \#A^c \leqslant \#\mathbb{N}\}$. (Note that $\#A \leqslant \#\mathbb{N}$ if, and only if, $\#A^c = \#\mathbb{R} \setminus A > \#\mathbb{N}$.)

$\underline{(M_1)}$ Since $\emptyset$ contains no elements, it is certainly countable and so $\gamma(\emptyset) = 0$.

$\underline{(M_2)}$ Let $(A_j)_{j\in\mathbb{N}}$ be pairwise disjoint $\mathcal{A}$-sets. If all of them are countable, then $A := \bigcup_{j\in\mathbb{N}}$ is countable and we get

$$\gamma\left(\bigcup_{j\in\mathbb{N}} A_j\right) = \gamma(A) = 0 = \sum_{j\in\mathbb{N}} \gamma(A_j).$$

If at least one A_j is not countable, say for $j = j_0$, then $A \supset A_{j_0}$ is not countable and therefore $\gamma(A) = \gamma(A_{j_0}) = 1$. Assume we could find some other $j_1 \neq j_0$ such that A_{j_0}, A_{j_1} are not countable. Since $A_{j_0}, A_{j_1} \in \mathcal{A}$ we know that their complements $A_{j_0}^c, A_{j_1}^c$ are countable, hence $A_{j_0}^c \cup A_{j_1}^c$ is countable and, at the same time, $\in \mathcal{A}$. Because of this, $(A_{j_0}^c \cup A_{j_1}^c)^c = A_{j_0} \cap A_{j_1} = \emptyset$ cannot be countable, which is absurd! Therefore there is at most one index $j_0 \in \mathbb{N}$ such that A_{j_0} is uncountable and we get then

$$\gamma\left(\bigcup_{j\in\mathbb{N}} A_j\right) = \gamma(A) = 1 = 1 + 0 + 0 + \dots = \gamma(A_{j_0}) + \sum_{j\neq j_0} \gamma(A_j).$$

(iii) We have an arbitrary measurable space $(X, \mathcal{A})$ and the measure $|A| = \begin{cases} \#A, & \text{if } A \text{ is finite} \\ \infty, & \text{else} \end{cases}$.

$\underline{(M_1)}$ Since $\emptyset$ contains no elements, $\#\emptyset = 0$ and $|\emptyset| = 0$.

$\underline{(M_2)}$ Let $(A_j)_{j\in\mathbb{N}}$ be a sequence of pairwise disjoint sets in $\mathcal{A}$. Case 1: All A_j are finite and only finitely many, say the first k, are non-empty, then $A = \bigcup_{j\in\mathbb{N}} A_j$ is effectively a finite union of k finite sets and it is clear that

$$|A| = |A_1| + \ldots + |A_k| + |\emptyset| + |\emptyset| + \ldots = \sum_{j\in\mathbb{N}} |A_j|.$$

Case 2: All A_j are finite and infinitely many are non-void. Then their union $A = \bigcup_{j\in\mathbb{N}} A_j$ is an infinite set and we get

$$|A| = \infty = \sum_{j\in\mathbb{N}} |A_j|.$$

Case 3: At least one A_j is infinite, and so is then the union $A = \bigcup_{j\in\mathbb{N}} A_j$. Thus,

$$|A| = \infty = \sum_{j\in\mathbb{N}} |A_j|.$$

(iv) On a countable set $\Omega = \{\omega_1, \omega_2, \ldots\}$ we define for a sequence $(p_j)_{j\in\mathbb{N}} \subset [0,1]$ with $\sum_{j\in\mathbb{N}} p_j = 1$ the set-function

$$P(A) = \sum_{j\,:\,\omega_j\in A} p_j = \sum_{j\in\mathbb{N}} p_j\, \delta_{\omega_j}(A), \qquad A \subset \Omega.$$

$\underline{(M_1)}$ $P(\emptyset) = 0$ is obvious.

$\underline{(M_2)}$ Let $(A_k)_{k\in\mathbb{N}}$ be pairwise disjoint subsets of Ω. Then

$$\begin{aligned}\sum_{k\in\mathbb{N}} P(A_k) &= \sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}} p_j\, \delta_{\omega_j}(A_k) \\ &= \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}} p_j\, \delta_{\omega_j}(A_k) \\ &= \sum_{j\in\mathbb{N}} p_j \Bigg(\sum_{k\in\mathbb{N}} \delta_{\omega_j}(A_k) \Bigg)\end{aligned}$$

$$= \sum_{j\in\mathbb{N}} p_j \delta_{\omega_j}\big(\textstyle\bigcup_k A_k\big)$$
$$= P\big(\textstyle\bigcup_k A_k\big).$$

The change in the order of summation needs justification; one possibility is the argument used in the solution of Problem 4.6(ii). (Note that the reordering theorem for absolutely convergent series is not immediately applicable since we deal with a double series!)

(v) This is obvious.

Problem 4.3 • On $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ the function γ is not be a measure, since we can take the sets $A = (1,\infty)$, $B = (-\infty,-1)$ which are disjoint, not countable and both have non-countable complements. Hence, $\gamma(A) = \gamma(B) = 1$. On the other hand, $A \uplus B$ is non-countable and has non-countable complement, $[-1,1]$. So, $\gamma(A \uplus B) = 1$. This contradicts the additivity: $\gamma(A \uplus B) = 1 \neq 2 = \gamma(A) + \gamma(B)$. *Notice that the choice of the σ-algebra $\mathcal{A}$ avoids exactly this situation. $\mathcal{B}$ is the wrong σ-algebra for γ.*

• On $\mathbb{Q}$ (and, actually, any possible σ-algebra thereon) the problem is totally different: if A is countable, then $A^c = \mathbb{Q} \setminus A$ is also countable and vice versa. This means that $\gamma(A)$ is, according to the definition, both 1 and 0 which is, of course, impossible. This is to say: γ is not well-defined. *γ makes only sense on a non-countable set X.*

Problem 4.4 (i) If $\mathcal{A} = \{\emptyset, \mathbb{R}\}$, then μ is a measure.

But as soon as $\mathcal{A}$ contains one set A which is trivial (i.e. either $\emptyset$ or X), we have actually $A^c \in \mathcal{A}$ which is also non-trivial. Thus,

$$1 = \mu(X) = \mu(A \uplus A^c) \neq \mu(A) + \mu(A^c) = 1 + 1 = 2$$

and μ cannot be a measure.

(ii) If we equip $\mathbb{R}$ with a σ-algebra which contains sets such that both A and A^c can be infinite (the Borel σ-algebra would be such an example: $A = (-\infty, 0) \implies A^c = [0,\infty)$), then ν is not well-defined. The only type of sets where ν is well-defined is, thus,

$$\mathcal{A} := \{A \subset \mathbb{R} \;:\; \#A < \infty \;\text{ or }\; \#A^c < \infty\}.$$

But this is no σ-algebra as the following example shows: $A_j := \{j\} \in \mathcal{A}$, $j \in \mathbb{N}$, are pairwise disjoint sets but $\bigcup_{j\in\mathbb{N}} A_j = \mathbb{N}$ is not finite and its complement is $\mathbb{R}\setminus\mathbb{N}$ not finite either! Thus, $\mathbb{N}\notin\mathcal{A}$, showing that $\mathcal{A}$ cannot be a σ-algebra. We conclude that ν can never be a measure if the σ-algebra contains infinitely many sets. If we are happy with finitely many sets only, then here is an example that makes ν into a measure $\mathcal{A}=\{\emptyset,\{69\},\mathbb{R}\setminus\{69\},\mathbb{R}\}$ and similar families are possible, but the point is that they all contain only finitely many members.

Problem 4.5 Denote by λ one-dimensional Lebesgue measure and consider the Borel sets $B_k := (k,\infty)$. Clearly $\bigcap_k B_k = \emptyset$, $k\in\mathbb{N}$, so that $B_k\downarrow\emptyset$. On the other hand,

$$\lambda(B_k)=\infty \implies \inf_k \lambda(B_k)=\infty\neq 0=\lambda(\emptyset)$$

which shows that the finiteness condition in Theorem 4.4 (iii′) and (iii″) is essential.

Problem 4.6 (i) Clearly, $\rho := a\mu+b\nu : \mathcal{A}\to[0,\infty]$ (since $a,b\geqslant 0$!). We check $(M_1),(M_2)$.

$\underline{(M_1)}$ Clearly, $\rho(\emptyset)=a\mu(\emptyset)+b\nu(\emptyset)=a\cdot 0+b\cdot 0=0$.

$\underline{(M_2)}$ Let $(A_j)_{j\in\mathbb{N}}\subset\mathcal{A}$ be mutually disjoint sets. Then we can use the σ-additivity of μ,ν to get

$$\begin{aligned}\rho\Big(\bigcup_{j\in\mathbb{N}}A_j\Big) &= a\mu\Big(\bigcup_{j\in\mathbb{N}}A_j\Big)+b\nu\Big(\bigcup_{j\in\mathbb{N}}A_j\Big)\\ &= a\sum_{j\in\mathbb{N}}\mu(A_j)+b\sum_{j\in\mathbb{N}}\nu(A_j)\\ &= \sum_{j\in\mathbb{N}}\big(a\mu(A_j)+b\mu(A_j)\big)\\ &= \sum_{j\in\mathbb{N}}\rho(A_j).\end{aligned}$$

Since all quantities involved are positive and since we allow the value $+\infty$ to be attained, there are no convergence problems.

(ii) Since all α_j are positive, the sum $\sum_{j\in\mathbb{N}}\alpha_j\mu_j(A)$ is a sum of positive quantities and, allowing the value $+\infty$ to be attained, there is no convergence problem. Thus, $\mu:\mathcal{A}\to[0,\infty]$ is well-defined. Before we check $(M_1),(M_2)$ we prove the following

Lemma. *Let β_{ij}, $i,j \in \mathbb{N}$, be real numbers. Then*

$$\sup_{i\in\mathbb{N}} \sup_{j\in\mathbb{N}} \beta_{ij} = \sup_{j\in\mathbb{N}} \sup_{i\in\mathbb{N}} \beta_{ij}.$$

Proof. Observe that we have $\beta_{mn} \leqslant \sup_{j\in\mathbb{N}} \sup_{i\in\mathbb{N}} \beta_{ij}$ for all $m,n \in \mathbb{N}$. The right-hand side is independent of m and n and we may take the *sup* over all n

$$\sup_{n\in\mathbb{N}} \beta_{mn} \leqslant \sup_{j\in\mathbb{N}} \sup_{i\in\mathbb{N}} \beta_{ij} \qquad \forall m \in \mathbb{N}$$

and then, with the same argument, take the sup over all m

$$\sup_{m\in\mathbb{N}} \sup_{n\in\mathbb{N}} \beta_{mn} \leqslant \sup_{j\in\mathbb{N}} \sup_{i\in\mathbb{N}} \beta_{ij} \qquad \forall m \in \mathbb{N}.$$

The opposite inequality, '$\geqslant$', follows from the same argument with i and j interchanged. ∎

(M_1) We have $\mu(\emptyset) = \sum_{j\in\mathbb{N}} \alpha_j \mu_j(\emptyset) = \sum_{j\in\mathbb{N}} \alpha_j \cdot 0 = 0$.

(M_2) Take pairwise disjoint sets $(A_i)_{i\in\mathbb{N}} \subset \mathcal{A}$. Then we can use the σ-additivity of each of the μ_j's to get

$$\begin{aligned}
\mu\Big(\bigcup_{i\in\mathbb{N}} A_i\Big) &= \sum_{j\in\mathbb{N}} \alpha_j \mu_j \Big(\bigcup_{i\in\mathbb{N}} A_i\Big) \\
&= \lim_{N\to\infty} \sum_{j=1}^{N} \alpha_j \sum_{i\in\mathbb{N}} \mu_j(A_i) \\
&= \lim_{N\to\infty} \sum_{j=1}^{N} \alpha_j \lim_{M\to\infty} \sum_{i=1}^{M} \mu_j(A_i) \\
&= \lim_{N\to\infty} \lim_{M\to\infty} \sum_{j=1}^{N} \sum_{i=1}^{M} \alpha_j \mu_j(A_i) \\
&= \sup_{N\in\mathbb{N}} \sup_{M\in\mathbb{N}} \sum_{j=1}^{N} \sum_{i=1}^{M} \alpha_j \mu_j(A_i)
\end{aligned}$$

where we used that the limits are increasing limits, hence suprema. By our lemma:

$$\mu\Big(\bigcup_{i\in\mathbb{N}} A_i\Big) = \sup_{M\in\mathbb{N}} \sup_{N\in\mathbb{N}} \sum_{i=1}^{M} \sum_{j=1}^{N} \alpha_j \mu_j(A_i)$$

$$= \lim_{M\to\infty} \lim_{N\to\infty} \sum_{i=1}^{M} \sum_{j=1}^{N} \alpha_j \mu_j (A_i)$$
$$= \lim_{M\to\infty} \sum_{i=1}^{M} \sum_{j\in\mathbb{N}} \alpha_j \mu_j (A_i)$$
$$= \lim_{M\to\infty} \sum_{i=1}^{M} \mu (A_i)$$
$$= \sum_{i\in\mathbb{N}} \mu (A_i) .$$

Problem 4.7 Set $\nu(A) := \mu(A \cap F)$. We know, by assumption, that μ is a measure on $(X, \mathcal{A})$. We have to show that ν is a measure on $(X, \mathcal{A})$. Since $F \in \mathcal{A}$, we have $F \cap A \in \mathcal{A}$ for all $A \in \mathcal{A}$, so ν is well-defined. Moreover, it is clear that $\nu(A) \in [0, \infty]$. Thus, we only have to check

$\underline{(M_1)}$ $\nu(\emptyset) = \mu(\emptyset \cap F) = \mu(\emptyset) = 0$.

$\underline{(M_2)}$ Let $(A_j)_{j\in\mathbb{N}} \subset \mathcal{A}$ be a sequence of pairwise disjoint sets. Then also $(A_j \cap F)_{j\in\mathbb{N}} \subset \mathcal{A}$ are pairwise disjoint and we can use the σ-additivity of μ to get

$$\nu\Big(\bigcup_{j\in\mathbb{N}} A_j\Big) = \mu\Big(F \cap \bigcup_{j\in\mathbb{N}} A_j\Big) = \mu\Big(\bigcup_{j\in\mathbb{N}} (F \cap A_j)\Big)$$
$$= \sum_{j\in\mathbb{N}} \mu(F \cap A_j)$$
$$= \sum_{j\in\mathbb{N}} \nu(A_j).$$

Problem 4.8 Since P is a probability measure, $P(A_j^c) = 1 - P(A_j) = 0$. By σ-subadditivity,

$$P\Big(\bigcup_{j\in\mathbb{N}} A_j^c\Big) \leqslant \sum_{j\in\mathbb{N}} P(A_j^c), = 0$$

and we conclude that

$$P\Big(\bigcap_{j\in\mathbb{N}} A_j\Big) = 1 - P\Big(\Big[\bigcap_{j\in\mathbb{N}} A_j\Big]^c\Big) = 1 - P\Big(\bigcup_{j\in\mathbb{N}} A_j^c\Big) = 1 - 0 = 0.$$

Problem 4.9 Note that

$$\bigcup_j A_j \setminus \bigcup_k B_k = \bigcup_j \Big(A_j \setminus \underbrace{\bigcup_k B_k}_{\supset B_j \ \forall j} \Big) \subset \bigcup_j (A_j \setminus B_j)$$

Since $\bigcup_j B_j \subset \bigcup_j A_j$ we get from σ-subadditivity

$$\begin{aligned} \mu\Big(\bigcup_j A_j\Big) - \mu\Big(\bigcup_j B_j\Big) &= \mu\Big(\bigcup_j A_j \setminus \bigcup_k B_k\Big) \\ &\leqslant \mu\Big(\bigcup_j (A_j \setminus B_j)\Big) \\ &\leqslant \sum_j \mu(A_j \setminus B_j). \end{aligned}$$

Problem 4.10 (i) We have $\emptyset \in \mathcal{A}$ and $\mu(\emptyset) = 0$, thus $\emptyset \in \mathcal{N}_\mu$.

(ii) Since $M \in \mathcal{A}$ (this is essential in order to apply μ to M!) we can use the monotonicity of measures to get $0 \leqslant \mu(M) \leqslant \mu(N) = 0$, i.e. $\mu(M) = 0$ and $M \in \mathcal{N}_\mu$ follows.

(iii) Since all $N_j \in \mathcal{A}$, we get $N := \bigcup_{j\in\mathbb{N}} N_j \in \mathcal{A}$. By the σ-subadditivity of a measure we find

$$0 \leqslant \mu(N) = \mu\Big(\bigcup_{j\in\mathbb{N}} N_j\Big) \leqslant \sum_{j\in\mathbb{N}} \mu(N_j) = 0,$$

hence $\mu(N) = 0$ and so $N \in \mathcal{N}_\mu$.

Problem 4.11 (i) The one-dimensional Borel sets $\mathcal{B} := \mathcal{B}^1$ are defined as the smallest σ-algebra containing the open sets. Pick $x \in \mathbb{R}$ and observe that the open intervals $(x - \frac{1}{k}, x + \frac{1}{k})$, $k \in \mathbb{N}$, are all open sets and therefore $(x - \frac{1}{k}, x + \frac{1}{k}) \in \mathcal{B}$. Since a σ-algebra is stable under countable intersections we get $\{x\} = \bigcap_{k\in\mathbb{N}}(x-\frac{1}{k}, x+\frac{1}{k}) \in \mathcal{B}$. Using the monotonicity of measures and the definition of Lebesgue measure we find

$$0 \leqslant \lambda(\{x\}) \leqslant \lambda((x - \tfrac{1}{k}, x + \tfrac{1}{k})) = (x + \tfrac{1}{k}) - (x - \tfrac{1}{k}) = \tfrac{2}{k} \xrightarrow{k\to\infty} 0.$$

[Following the hint leads to a similar proof with $[x - \frac{1}{k}, x + \frac{1}{k})$ instead of $(x - \frac{1}{k}, x + \frac{1}{k})$.]

(ii) a) Since $\mathbb{Q}$ is countable, we find an enumeration $\{q_1, q_2, q_3, \ldots\}$ and we get trivially $\mathbb{Q} = \bigcup_{j\in\mathbb{N}}\{q_j\}$ which is a disjoint union. (This shows, by the way, that $\mathbb{Q} \in \mathcal{B}$ as $\{q_j\} \in \mathcal{B}$.) Therefore, using part (i) of the problem and the σ-additivity of measures,

$$\lambda(\mathbb{Q}) = \lambda\Big(\bigcup_{j\in\mathbb{N}}\{q_j\}\Big) = \sum_{j\in\mathbb{N}}\lambda(\{q_j\}) = \sum_{j\in\mathbb{N}} 0 = 0.$$

b) Take again an enumeration $\mathbb{Q} = \{q_1, q_2, q_3, \ldots\}$, fix $\epsilon > 0$ and define $C(\epsilon)$ as stated in the problem. Then we have $C(\epsilon) \in \mathcal{B}$ and $\mathbb{Q} \subset C(\epsilon)$. Using the monotonicity and σ-subadditivity of λ we get

$$\begin{aligned}
0 \leqslant \lambda(\mathbb{Q}) &\leqslant \lambda\big(C(\epsilon)\big)\\
&= \lambda\Big(\bigcup_{k\in\mathbb{N}}[q_k - \epsilon 2^{-k}, q_k + \epsilon 2^{-k})\Big)\\
&\leqslant \sum_{k\in\mathbb{N}}\lambda\big([q_k - \epsilon 2^{-k}, q_k + \epsilon 2^{-k})\big)\\
&= \sum_{k\in\mathbb{N}} 2\cdot\epsilon\cdot 2^{-k}\\
&= 2\epsilon\,\frac{\frac{1}{2}}{1-\frac{1}{2}} = 2\epsilon.
\end{aligned}$$

As $\epsilon > 0$ was arbitrary, we can make $\epsilon \to 0$ and the claim follows.

(iii) Since $\bigcup_{0\leqslant x\leqslant 1}\{x\}$ is a disjoint union, only the countability assumption is violated. Let's see what happens if we could use 'σ-additivity' for such non-countable unions:

$$0 = \sum_{0\leqslant x\leqslant 1} 0 = \sum_{0\leqslant x\leqslant 1}\lambda(\{x\}) = \lambda\Big(\bigcup_{0\leqslant x\leqslant 1}\{x\}\Big) = \lambda([0,1]) = 1$$

which is impossible.

Problem 4.12 Without loss of generality we may assume that $a \neq b$; set $\mu := \delta_a + \delta_b$. Then $\mu(B) = 0$ if, and only if, $a \notin B$ and $b \notin B$. Since $\{a\}, \{b\}$ and $\{a, b\}$ are Borel sets, all null sets of μ are given by

$$\mathcal{N}_\mu = \big\{B \setminus \{a,b\} \;:\; B \in \mathcal{B}(\mathbb{R})\big\}.$$

(This shows that, in some sense, null sets can be fairly large!).

Problem 4.13 Let us write $\mathfrak{N}$ for the family of all (proper and improper) subsets of μ null sets. We note that sets in $\mathfrak{N}$ can be measurable (that is: $N \in \mathcal{A}$) but need not be measurable.

(i) Since $\emptyset \in \mathfrak{N}$, we find that $A = A \cup \emptyset \in \mathcal{A}^*$ for every $A \in \mathcal{A}$; thus, $\mathcal{A} \subset \mathcal{A}^*$. Let us check that $\mathcal{A}^*$ is a σ-algebra.

(Σ_1) Since $\emptyset \in \mathcal{A} \subset \mathcal{A}^*$, we have $\emptyset \in \mathcal{A}^*$.

(Σ_2) Let $A^* \in \mathcal{A}^*$. Then $A^* = A \cup N$ for $A \in \mathcal{A}$ and $N \in \mathfrak{N}$. By definition, $N \subset M \in \mathcal{A}$ where $\mu(M) = 0$. Now

$$\begin{aligned} A^{*c} = (A \cup N)^c &= A^c \cap N^c \\ &= A^c \cap N^c \cap (M^c \cup M) \\ &= (A^c \cap N^c \cap M^c) \cup (A^c \cap N^c \cap M) \\ &= (A^c \cap M^c) \cup (A^c \cap N^c \cap M) \end{aligned}$$

where we used that $N \subset M$, hence $M^c \subset N^c$, hence $M^c \cap N^c = M^c$. But now we see that $A^c \cap M^c \in \mathcal{A}$ and $A^c \cap N^c \cap M \in \mathfrak{N}$ since $A^c \cap N^c \cap M \subset M$ and $M \in \mathcal{A}$ is a μ null set: $\mu(M) = 0$.

(Σ_3) Let $(A_j^*)_{j\in\mathbb{N}}$ be a sequence of $\mathcal{A}^*$-sets. From its very definition we know that each $A_j^* = A_j \cup N_j$ for some (not necessarily unique!) $A_j \in \mathcal{A}$ and $N_j \in \mathfrak{N}$. So,

$$\bigcup_{j\in\mathbb{N}} A_j^* = \bigcup_{j\in\mathbb{N}} (A_j \cup N_j) = \Big(\bigcup_{j\in\mathbb{N}} A_j\Big) \cup \Big(\bigcup_{j\in\mathbb{N}} N_j\Big) =: A \cup N.$$

Since $\mathcal{A}$ is a σ-algebra, $A \in \mathcal{A}$. All we have to show is that N_j is in $\mathfrak{N}$. Since each N_j is a subset of a (measurable!) null set, say, $M_j \in \mathcal{A}$, we find that $N = \bigcup_{j\in\mathbb{N}} N_j \subset \bigcup_{j\in\mathbb{N}} M_j = M \in \mathcal{A}$ and all we have to show is that $\mu(M) = 0$. But this follows from σ-subadditivity,

$$0 \leqslant \mu(M) = \mu\Big(\bigcup_{j\in\mathbb{N}} M_j\Big) \leqslant \sum_{j\in\mathbb{N}} \mu(M_j) = 0.$$

Thus, $A \cup N \in \mathcal{A}^*$.

(ii) As already mentioned in part (i), $A^* \in \mathcal{A}^*$ could have more than one representation, e.g. $A \cup N = A^* = B \cup M$ with $A, B \in \mathcal{A}$ and $N, M \in \mathfrak{N}$. If we can show that $\mu(A) = \mu(B)$ then the definition of $\bar{\mu}$ is independent of the representation of A^*. Since M, N are not necessarily measurable but, by definition, subsets of (measurable) null sets $M', N' \in \mathcal{A}$ we find

$$A \subset A \cup N = B \cup M \subset B \cup M',$$
$$B \subset B \cup M = A \cup N \subset A \cup N'$$

and since $A, B, B \cup M', A \cup N' \in \mathcal{A}$, we get from monotonicity and subadditivity of measures

$$\mu(A) \leqslant \mu(B \cup M') \leqslant \mu(B) + \mu(M') = \mu(B),$$
$$\mu(B) \leqslant \mu(A \cup N') \leqslant \mu(A) + \mu(N') = \mu(A)$$

which shows $\mu(A) = \mu(B)$.

(iii) We check (M_1) and (M_2)

(M_1) Since $\emptyset = \emptyset \cup \emptyset \in \mathcal{A}^*$, $\emptyset \in \mathcal{A}$, $\emptyset \in \mathfrak{N}$, we have $\bar{\mu}(\emptyset) = \mu(\emptyset) = 0$.

(M_2) Let $(A_j^*)_{j\in\mathbb{N}} \subset \mathcal{A}^*$ be a sequence of pairwise disjoint sets. Then $A_j^* = A_j \cup N_j$ for some $A_j \in \mathcal{A}$ and $N_j \in \mathfrak{N}$. These sets are also mutually disjoint, and with the arguments in (i) we see that $A^* = A \cup N$ where $A^* \in \mathcal{A}^*, A \in \mathcal{A}, N \in \mathfrak{N}$ stand for the unions of A_j^*, A_j and N_j, respectively. Since $\bar{\mu}$ does not depend on the special representation of $\mathcal{A}^*$-sets, we get

$$\begin{aligned}\bar{\mu}\Big(\bigcup_{j\in\mathbb{N}} A_j^*\Big) = \bar{\mu}(A^*) = \mu(A) &= \mu\Big(\bigcup_{j\in\mathbb{N}} A_j\Big)\\ &= \sum_{j\in\mathbb{N}} \mu(A_j)\\ &= \sum_{j\in\mathbb{N}} \bar{\mu}(A_j^*)\end{aligned}$$

showing that $\bar{\mu}$ is σ-additive.

(iv) Let M^* be a $\bar{\mu}$ null set, i.e. $M^* \in \mathcal{A}^*$ and $\bar{\mu}(M^*) = 0$. Take any $B \subset M^*$. We have to show that $B \in \mathcal{A}^*$ and $\bar{\mu}(B) = 0$. The latter is clear from the monotonicity of $\bar{\mu}$ once we have shown that $B \in \mathcal{A}^*$ which means, once we know that we may plug B into $\bar{\mu}$.

Now, $B \subset M^*$ and $M^* = M \cup N$ for some $M \in \mathcal{A}$ and $N \in \mathfrak{N}$. As $\bar{\mu}(M^*) = 0$ we also know that $\mu(M) = 0$. Moreover, we know from the definition of $\mathfrak{N}$ that $N \subset N'$ for some $N' \in \mathcal{A}$ with $\mu(N') = 0$. This entails

$$B \subset M^* = M \cup N \subset M \cup N' \in \mathcal{A}$$

$$\text{and} \quad \mu(M \cup N') \leqslant \mu(M) + \mu(N') = 0.$$

Hence $B \in \mathfrak{N}$ as well as $B = \emptyset \cup B \in \mathcal{A}^*$. In particular, $\bar{\mu}(B) = \mu(\emptyset) = 0$.

(v) Set $\mathcal{C} = \{A^* \subset X : \exists A, B \in \mathcal{A}, \quad A \subset A^* A \subset B, \quad \mu(B \setminus A) = 0\}$. We have to show that $\mathcal{A}^* = \mathcal{C}$.

Take $A^* \in \mathcal{A}^*$. Then $A^* = A \cup N$ with $A \in \mathcal{A}$, $N \in \mathfrak{N}$ and choose $N' \in \mathcal{A}$, $N \subset N'$ and $\mu(N') = 0$. This shows that

$$A \subset A^* = A \cup N \subset A \cup N' =: B \in \mathcal{A}$$

and that $\mu(B \setminus A) = \mu((A \cup N') \setminus A) \leqslant \mu(N') = 0$. (Note that $(A \cup N') \setminus A = (A \cup N') \cap A^c = N' \cap A^c \subset N'$ and that equality need not hold!).

Conversely, take $A^* \in \mathcal{C}$. Then, by definition, $A \subset A^* \subset B$ with $A, B \in \mathcal{A}$ and $\mu(B \setminus A) = 0$. Therefore, $N := B \setminus A$ is a null set and we see that $A^* \setminus A \subset B \setminus A$, i.e. $A^* \setminus A \in \mathfrak{N}$. So, $A^* = A \cup (A^* \setminus A)$ where $A \in \mathcal{A}$ and $A^* \setminus A \in \mathfrak{N}$ showing that $A^* \in \mathcal{A}^*$.

Problem 4.14 (i) Since $\mathcal{B}$ is a σ-algebra, it is closed under countable (disjoint) unions of its elements, thus ν inherits the properties (M_1), (M_2) directly from μ.

(ii) Yes [yes], since the full space $X \in \mathcal{B}$ so that $\mu(X) = \nu(X)$ is finite [resp. $= 1$].

(iii) No, σ-finiteness is also a property of the σ-algebra. Take, for example, Lebesgue measure λ on the Borel sets (this is σ-finite) and consider the σ-algebra $\mathcal{C} := \{\emptyset, (-\infty, 0), [0, \infty), \mathbb{R}\}$. Then $\lambda\big|_{\mathcal{C}}$ is not σ-finite since there is no increasing sequence of $\mathcal{C}$-sets having finite measure.

Problem 4.15 By definition, μ is σ-finite if there is an *increasing* sequence $(B_j)_{j \in \mathbb{N}} \subset \mathcal{A}$ such that $B_j \uparrow X$ and $\mu(B_j) < \infty$. Clearly, $E_j := B_j$ satisfies the condition in the statement of the problem.

Conversely, let $(E_j)_{j \in \mathbb{N}}$ be as stated in the problem. Then $B_n := E_1 \cup \ldots \cup E_n$ is measurable, $B_n \uparrow X$ and, by subadditivity,

$$\mu(B_n) = \mu(E_1 \cup \ldots \cup E_n) \leqslant \sum_{j=1}^{n} \mu(E_j) < \infty.$$

Remark: A small change in the above argument allows to take pairwise disjoint sets E_j.