5 Uniqueness of measures. Solutions to Problems 5.1–5.10

Problem 5.1 Since $X \in \mathcal{D}$ and since complements are again in $\mathcal{D}$, we have $\emptyset = X^c \in \mathcal{D}$.

If $A, B \in \mathcal{D}$ are disjoint, we set $A_1 := A, A_2 := B, A_j := \emptyset\ \forall j \geqslant 3$. Then $(A_j)_{j\in\mathbb{N}} \subset \mathcal{D}$ is a sequence of pairwise disjoint sets, and by (Δ_3) we find that

$$A \mathbin{\dot\cup} B = \dot{\bigcup_{j\in\mathbb{N}}} A_j \in \mathcal{D}.$$

Since $(\Sigma_1) = (\Delta_3)$, $(\Sigma_2) = (\Delta_2)$ and since $(\Sigma_3) \implies (\Delta_3)$, it is clear that every σ-algebra is also a Dynkin system; that the converse is, in general, wrong is seen in Problem 5.2.

Problem 5.2 Consider (Δ_3) only, as the other two conditions coincide: $(\Sigma_j) = (\Delta_j)$, $j = 1, 2$. We show that (Σ_3) breaks down even for finite unions. If $A, B \in \mathcal{D}$ are disjoint, it is clear that A, B *and also* $A \mathbin{\dot\cup} B$ contain an even number of elements. But if A, B have non-void intersection, and if this intersection contains an odd number of elements, then $A \cup B$ contains an odd number of elements. Here is a trivial example:

$$A = \{1, 2\} \in \mathcal{D}, \quad B = \{2, 3, 4, 5\} \in \mathcal{D},$$

whereas

$$A \cup B = \{1, 2, 3, 4, 5\} \notin \mathcal{D}.$$

This means that (Δ_3) holds, but (Σ_3) fails.

Problem 5.3 *Mind the* ***misprint:*** *$A \subset B$ must be assumed and is* ***missing*** *in the statement of the problem!* We verify the hint first. Using de Morgan's laws we get

$$R \setminus Q = R \setminus (R \cap Q) = R \cap (R \cap Q)^c = (R^c \cup (R \cap Q))^c = (R^c \mathbin{\dot\cup} (R \cap Q))^c$$

where the last equality follows since $R^c \cap (R \cap Q) = \emptyset$.

Now we take $A, B \in \mathcal{D}$ such that $A \subset B$. In particular $A \cap B = A$. Taking this into account and setting $Q = A, R = B$ we get from the above relation

$$B \setminus A = \big(\underbrace{\underbrace{B^c}_{\in \mathcal{D}} \cupdot A}_{\in \mathcal{D}} \big)^c \in \mathcal{D}$$

where we repeatedly use (Δ_2) and (Δ_2).

Problem 5.4 (i) Since the σ-algebra $\mathcal{A}$ is also a Dynkin system, it is enough to prove $\delta(\mathcal{D}) = \mathcal{D}$ for any Dynkin system $\mathcal{D}$. By definition, $\delta(\mathcal{D})$ is the smallest Dynkin system containing $\mathcal{D}$, thus $\mathcal{D} \subset \delta(\mathcal{D})$. On the other hand, $\mathcal{D}$ is itself a Dynkin system, thus, because of minimality, $\mathcal{D} \supset \delta(\mathcal{D})$.

(ii) Clearly, $\mathcal{G} \subset \mathcal{H} \subset \delta(\mathcal{H})$. Since $\delta(\mathcal{H})$ is a Dynkin system containing $\mathcal{G}$, the minimality of $\delta(\mathcal{G})$ implies that $\delta(\mathcal{G}) \subset \delta(\mathcal{H})$.

(iii) Since $\sigma(\mathcal{G})$ is a σ-algebra, it is also a Dynkin system. Since $\mathcal{G} \subset \sigma(\mathcal{G})$ we conclude (again, by minimality) that $\delta(\mathcal{G}) \subset \sigma(\mathcal{G})$.

Problem 5.5 Clearly, $\delta(\{A, B\}) \subset \sigma(\{A, B\})$ is always true.

By Theorem 5.5, $\delta(\{A, B\}) = \sigma(\{A, B\})$ if $\{A, B\}$ is $\cap$-stable, i.e. if $A = B$ or $A = B^c$ or if at least one of A, B is X or $\emptyset$.

Let us exclude these cases. If $A \cap B = \emptyset$, then

$$\delta(\{A, B\}) = \sigma(\{A, B\}) = \big\{\emptyset, A, A^c, B, B^c, A \cupdot B, A^c \cap B^c, X\big\}.$$

If $A \cap B \neq \emptyset$, then

$$\delta(\{A, B\}) = \big\{\emptyset, A, A^c, B, B^c, X\big\}$$

while $\sigma(\{A, B\})$ is much larger containing, for example, $A \cap B$.

Problem 5.6 We prove the hint first. Let $(G_j)_{j\in\mathbb{N}} \subset \mathcal{G}$ as stated in the problem, i.e. satisfying (1) and (2), and define the sets $F_N := G_1 \cup \ldots \cup G_N$. As $\mathcal{G} \subset \mathcal{A}$, it is clear that $F_N \in \mathcal{A}$ (but not necessarily in $\mathcal{G}$...). Moreover, it is clear that $F_N \uparrow X$.

We begin with a more general assertion: *For any finite union of $\mathcal{G}$-sets $A_1 \cup \ldots \cup A_N$ we have $\mu(A_1 \cup \ldots \cup A_N) = \nu(A_1 \cup \ldots \cup A_N)$.*

Proof. *Induction Hypothesis:* $\mu(A_1 \cup \ldots \cup A_N) = \nu(A_1 \cup \ldots \cup A_N)$ for some $N \in \mathbb{N}$ and any choice of $A_1, \ldots, A_N \in \mathcal{G}$.

Induction Start $(N = 1)$*:* is obvious.

Induction Step $N \rightsquigarrow N+1$: We have by the strong additivity of measures and the $\cap$-stability of $\mathcal{G}$ that

$$\begin{aligned}
&\mu\big(A_1 \cup \ldots \cup A_N \cup A_{N+1}\big) \\
&= \mu\big((A_1 \cup \ldots \cup A_N) \cup A_{N+1}\big) \\
&= \mu\big(A_1 \cup \ldots \cup A_N\big) + \mu(A_{N+1}) - \mu\big((A_1 \cup \ldots \cup A_N) \cap A_{N+1}\big) \\
&= \mu\big(A_1 \cup \ldots \cup A_N\big) + \mu(A_{N+1}) - \mu\big(\underbrace{(A_1 \cap A_{N+1})}_{\in \mathcal{G}} \cup \ldots \cup \underbrace{(A_N \cap A_{N+1})}_{\in \mathcal{G}}\big) \\
&= \nu\big(A_1 \cup \ldots \cup A_N\big) + \nu(A_{N+1}) - \nu\big((A_1 \cap A_{N+1}) \cup \ldots \cup (A_N \cap A_{N+1})\big) \\
&\ \vdots \\
&= \nu\big(A_1 \cup \ldots \cup A_N \cup A_{N+1}\big)
\end{aligned}$$

where we used the induction hypothesis twice, namely for the union of the N $\mathcal{G}$-sets $A_1, \ldots, A_N$ as well as for the N $\mathcal{G}$-sets $A_1 \cap A_{N+1}, \ldots, A_N \cap A_{N+1}$. The induction is complete.

In particular we see that $\mu(F_N) = \nu(F_N)$, $\nu(F_N) \leqslant \nu(G_1) + \ldots + \nu(G_N) < \infty$ by subadditivity, and that (think!) $\mu(G \cap F_N) = \nu(G \cap F_N)$ for any $G \in \mathcal{G}$ (just work out the intersection, similar to the step in the induction....). This shows that on the $\cap$-stable system

$$\tilde{\mathcal{G}} := \{\text{all finite unions of sets in} \quad \mathcal{G}\}$$

μ and ν coincide. Moreover, $\mathcal{G} \subset \tilde{\mathcal{G}} \subset \mathcal{A}$ so that, by assumption $\mathcal{A} = \sigma(\mathcal{G}) \subset \sigma(\tilde{\mathcal{G}}) \subset \sigma(\mathcal{A}) \subset \mathcal{A}$, so that equality prevails in this chain of inclusions. This means that $\tilde{\mathcal{G}}$ is a generator of $\mathcal{A}$ satisfying all the assumptions of Theorem 5.7, and we have reduced everything to this situation.

Problem 5.7 *Intuition:* in two dimensions we have rectangles. Take $I, I' \in \mathcal{J}$. Call the lower left corner of I $a = (a_1, a_2)$, the upper right corner $b = (b_1, b_2)$, and do the same for I' using a', b'. This defines a rectangle uniquely. We are done, if $I \cap I' = \emptyset$. If not (draw a picture!) then we get an overlap which can be described by taking the right-and-upper-most of the two lower left corners a, a' and the left-and-lower-most of the two upper right corners b, b'. That does the trick.

Now rigorously: since $I, I' \in \mathcal{J}$, we have for suitable a_j, b_j, a'_j, b'_j's:

$$I = \mathop{\times}_{j=1}^{n} \big[a_j, b_j\big) \quad \text{and} \quad I' = \mathop{\times}_{j=1}^{n} \big[a'_j, b'_j\big).$$

We want to find $I \cap I'$, or, equivalently the condition under which $x \in I \cap I'$. Now

$$\begin{aligned} x = (x_1, \dots, x_n) \in I &\iff x_j \in [a_j, b_j) \quad \forall j = 1, 2, \dots, n \\ &\iff a_j \leqslant x_j < b_j \quad \forall j = 1, 2, \dots, n \end{aligned}$$

and the same holds for $x \in I'$ (same x, but I'—no typo). Clearly $a_j \leqslant x_j < b_j$, and, at the same time $a'_j \leqslant x_j < b'_j$ holds exactly if

$$\begin{aligned} &\max(a_j, a'_j) \leqslant x_j < \min(b_j, b'_j) \quad \forall j = 1, 2, \dots, n \\ \iff & x \in \mathop{\times}_{j=1}^{n} \big[\max(a_j, a'_j), \min(b_j, b'_j)\big). \end{aligned}$$

This shows that $I \cap I'$ is indeed a 'rectangle', i.e. in $\mathcal{J}$. This could be an empty set (which happens if I and I' do not meet).

Problem 5.8 First we must make sure that $t \cdot B$ is a Borel set if $B \in \mathcal{B}$. We consider first rectangles $I = [\![a, b)\!) \in \mathcal{J}$ where $a, b \in \mathbb{R}^n$. Clearly, $t \cdot I = [\![ta, tb)\!)$ where ta, tb are just the scaled vectors. So, scaled rectangles are again rectangles, and therefore Borel sets. Now fix $t > 0$ and set

$$\mathcal{B}_t := \{B \in \mathcal{B}^n \ : \ t \cdot B \in \mathcal{B}^n\}.$$

It is not hard to see that $\mathcal{B}_t$ is itself a σ-algebra and that $\mathcal{J} \subset \mathcal{B}_t \subset \mathcal{B}^n$. But then we get

$$\mathcal{B}^n = \sigma(\mathcal{J}) \subset \sigma(\mathcal{B}_t) = \mathcal{B}_t \subset \mathcal{B}^n,$$

showing that $\mathcal{B}_t = \mathcal{B}^n$, i.e. scaled Borel sets are again Borel sets.

Now define a new measure $\mu(B) := \lambda^n(t \cdot B)$ for Borel sets $B \in \mathcal{B}^n$ (which is, because of the above, well-defined). For rectangles $[\![a, b)\!)$ we get, in particular,

$$\begin{aligned} \mu[\![a, b)\!) = \lambda^n\big((t \cdot [\![a, b)\!)\big) &= \lambda^n[\![ta, tb)\!) \\ &= \prod_{j=1}^{n} \big((tb_j) - (ta_j)\big) \\ &= \prod_{j=1}^{n} t \cdot \big(b_j - a_j\big) \\ &= t^n \cdot \prod_{j=1}^{n} \big(b_j - a_j\big) \end{aligned}$$

$$= t^n \lambda^n [\![a, b)\!)$$

which shows that μ and $t^n\lambda^n$ coincide on the $\cap$-stable generator $\mathcal{J}$ of $\mathcal{B}^n$, hence they're the same everywhere. (Mind the small gap: we should make the mental step that for any measure ν a positive multiple, say, $c \cdot \nu$, is again a measure—this ensures that $t^n\lambda^n$ is a measure, and we need this in order to apply Theorem 5.7. Mind also that we need that μ is finite on all rectangles (obvious!) and that we find rectangles increasing to $\mathbb{R}^n$, e.g. $[-k, k) \times \ldots \times [-k, k)$ as in the proof of Theorem 5.8(ii).)

Problem 5.9 Define $\nu(A) := \mu \circ \theta^{-1}(A)$. Obviously, ν is again a finite measure. Moreover, since $\theta^{-1}(X) = X$, we have

$$\mu(X) = \nu(X) < \infty \quad \text{and, by assumption,} \quad \mu(G) = \nu(G) \quad \forall\, G \in \mathcal{G}.$$

Thus, $\mu = \nu$ on $\mathcal{G}' := \mathcal{G} \cup \{X\}$. Since $\mathcal{G}'$ is a $\cap$-stable generator of $\mathcal{A}$ containing the (trivial) exhausting sequence $X, X, X, \ldots$, the assertion follows from the uniqueness theorem for measures, Theorem 5.7.

Problem 5.10 The necessity of the condition is trivial since $\mathcal{G} \subset \sigma(\mathcal{G}) = \mathcal{B}$, resp., $\mathcal{H} \subset \sigma(\mathcal{H}) = \mathcal{C}$.

Fix $H \in \mathcal{H}$ and define

$$\mu(B) := P(B \cap H) \quad \text{and} \quad \nu(B) := P(B)P(H).$$

Obviously, μ and ν are finite measures on $\mathcal{B}$ having mass $P(H)$ such that μ and ν coincide on the $\cap$-stable generator $\mathcal{G} \cup \{X\}$ of $\mathcal{B}$. Note that this generator contains the exhausting sequence $X, X, X, \ldots$. By the uniqueness theorem for measures, Theorem 5.7, we conclude

$$\mu = \nu \quad \text{on the whole of} \quad \mathcal{B}.$$

Now fix $B \in \mathcal{B}$ and define

$$\rho(C) := P(B \cap C) \quad \text{and} \quad \tau(C) := P(B)P(C).$$

Then the same argument as before shows that $\rho = \sigma$ on $\mathcal{C}$ and, since $B \in \mathcal{B}$ was arbitrary, the claim follows.

6 Existence of measures. Solutions to Problems 6.1–6.11

Problem 6.1 We know already that $\mathcal{B}[0,\infty)$ is a σ-algebra (it is a trace σ-algebra) and, by definition,

$$\Sigma = \big\{B \cup (-B) \;:\; B \in \mathcal{B}[0,\infty)\big\}$$

if we write $-B := \{-b \;:\; b \in \mathcal{B}[0,\infty)\}$.

Since the structure $B \cup (-B)$ is stable under complementation and countable unions it is clear that Σ is indeed a σ-algebra.

One possibility to extend μ defined on Σ would be to take $B \in \mathcal{B}(\mathbb{R})$ and define $B^+ := B \cap [0,\infty)$ and $B^- := B \cap (-\infty, 0)$ and to set

$$\nu(B) := \mu(B^+ \cup (-B^+)) + \mu((-B^-) \cup B^-)$$

which is obviously a measure. We cannot expect uniqueness of this extension since Σ does not generate $\mathcal{B}(\mathbb{R})$—not all Borel sets are symmetric.

Problem 6.2 By definition we have

$$\mu^*(Q) = \inf\left\{\sum_j \mu(B_j) \;:\; (B_j)_{j\in\mathbb{N}} \subset \mathcal{A},\ \bigcup_{j\in\mathbb{N}} B_j \supset Q\right\}.$$

(i) Assume first that $\mu^*(Q) < \infty$. By the definition of the infimum we find for every $\epsilon > 0$ a sequence $(B_j^\epsilon)_{j\in\mathbb{N}} \subset \mathcal{A}$ such that $B^\epsilon := \bigcup_j B_j^\epsilon \supset Q$ and, because of σ-subadditivity,

$$\mu(B^\epsilon) - \mu^*(Q) \leqslant \sum_j \mu(B_j^\epsilon) - \mu^*(Q) \leqslant \epsilon.$$

Set $B := \bigcap_k B^{1/k} \in \mathcal{A}$. Then $B \supset Q$ and $\mu(B) = \mu^*(B) = \mu^*(Q)$. By the very definition of $\mathcal{A}^*$ and since $B \in \mathcal{A} \subset \mathcal{A}^*$ we get

$$\mu^*(Q) \overset{(6.4)}{=} \mu^*(B \cap Q) + \mu^*(B \setminus Q) = \mu(B) + \mu^*(B \setminus Q)$$

so that $\mu^*(B \setminus Q) = 0$. Since (the outer measure) μ^* is monotone, we conclude that for all $\mathcal{A}$-measurable sets $N \subset A \setminus Q$ we have $\mu(N) = \mu^*(N) \leqslant \mu^*(B \setminus Q) = 0$.

If $\mu^*(Q) = \infty$, we take the exhausting sequence $(A_j)_{j\in\mathbb{N}} \subset \mathcal{A}$ with $A_j \uparrow X$ and $\mu(A_j) < \infty$ and set $Q_j := A_j \cap Q$ for every $j \in \mathbb{N}$. By the first part we can find sets $C_j \in \mathcal{A}$ with $\mu(C_j) = \mu^*(Q_j)$ and $\mu^*(C_j \setminus Q_j) = 0$. Then

$$C := \bigcup_j C_j \supset \bigcup_j Q_j = Q, \quad \mu(C) = \infty = \mu^*(Q)$$

and, using (the hint of) Problem 4.9, and the monotonicity and σ-subadditivity of μ^*:

$$\bigcup_j C_j \Big\backslash \bigcup_j Q_j \subset \bigcup_j C_j \setminus Q_j$$

and

$$\mu^*\Big(\bigcup_j C_j \Big\backslash \bigcup_j Q_j\Big) \leqslant \mu^*\Big(\bigcup_j C_j \setminus Q_j\Big) \leqslant \sum_j \mu^*\big(\bigcup_j C_j \setminus Q_j\big) = 0.$$

(ii) Define $\bar\mu := \mu^*\big|_{\mathcal{A}^*}$. We know from Theorem 6.1 that $\bar\mu$ is a measure on $\mathcal{A}^*$ and, because of the monotonicity of μ^*, we know that for all $N^* \in \mathcal{A}^*$ with $\bar\mu(N^*)$ we have

$$\forall\, M \subset N^* \ :\ \mu^*(M) \leqslant \mu^*(N^*) = \bar\mu(N^*) = 0.$$

It remains to show that $M \in \mathcal{A}^*$. Because of (6.4) we have to show that

$$\forall\, Q \subset X \ :\ \mu^*(Q) = \mu^*(Q \cap M) + \mu(Q \setminus M).$$

Since μ^* is subadditive we find for all $Q \subset X$

$$\begin{aligned} \mu^*(Q) &= \mu^*\big((Q \cap M) \cup (Q \setminus M)\big) \\ &\leqslant \mu^*(Q \cap M) + \mu^*(Q \setminus M) \\ &= \mu^*(Q \setminus M) \\ &\leqslant \mu^*(Q), \end{aligned}$$

which means that $M \in \mathcal{A}^*$.

(iii) Obviously, $(X, \mathcal{A}^*, \bar\mu)$ extends $(X, \mathcal{A}, \mu)$ since $\mathcal{A} \subset \mathcal{A}^*$ and $\bar\mu\big|_{\mathcal{A}} = \mu$. In view of Problem 4.13 we have to show that

$$\mathcal{A}^* = \{A \cup N \ :\ A \in \mathcal{A}, \quad N \in \mathfrak{N}\} \qquad (*)$$

with $\mathfrak{N} = \{N \subset X : N$ is subset of an $\mathcal{A}$-measurable null set or, alternatively,

$$\mathcal{A}^* = \{A^* \subset X : \exists\, A, B \in \mathcal{A},\ A \subset A^* \subset B,\ \mu(B \setminus A) = 0\}. \quad (**)$$

We are going to use both equalities and show '$\supset$' in $(*)$ and '$\subset$' in $(**)$ (which is enough since, cf. Problem 4.13 asserts the equality of the right-hand sides of $(*), (**)!$).

<u>'$\supset$':</u> By part (ii), subsets of $\mathcal{A}$-null sets are in $\mathcal{A}^*$ so that every set of the form $A \cup N$ with $A \in \mathcal{A}$ and N being a subset of an $\mathcal{A}$ null set is in $\mathcal{A}^*$.

<u>'$\subset$':</u> By part (i) we find for every $A^* \in \mathcal{A}^*$ some $A \in \mathcal{A}$ such that $A \supset A^*$ and $A \setminus A^*$ is an $\mathcal{A}^*$ null set. By the same argument we get $B \in \mathcal{A}$, $B \supset (A^*)^c$ and $B \setminus (A^*)^c = B \cap A^* = A^* \setminus B^c$ is an $\mathcal{A}^*$ null set. Thus,

$$B^c \subset A^* \subset A$$

and

$$A \setminus B^c \subset \big(A \setminus A^*\big) \cup \big(A^* \setminus B^c\big) = \big(A \setminus A^*\big) \cup \big(B \setminus (A^*)^c\big)$$

which is the union of two $\mathcal{A}^*$ null sets, i.e. $A \setminus B^c$ is an $\mathcal{A}$ null set.

Problem 6.3 (i) A little geometry first: a solid, open disk of radius r, centre 0 is the set $B_r(0) := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < r^2\}$. Now the n-dimensional analogue is clearly $\{x \in \mathbb{R}^n : x_1^2 + x_2^2 + \ldots + x_n^2 < r^2\}$ (including $n = 1$ where it reduces to an interval). We want to inscribe a box into a ball.

Claim: $Q_\epsilon(0) := \bigtimes_{j=1}^n \left[-\frac{\epsilon}{\sqrt{n}}, \frac{\epsilon}{\sqrt{n}}\right) \subset B_{2\epsilon}(0)$. Indeed,

$$\begin{aligned} x \in Q_\epsilon(0) &\implies x_1^2 + x_2^2 + \ldots + x_n^2 \leqslant \frac{\epsilon^2}{n} + \frac{\epsilon^2}{n} + \ldots + \frac{\epsilon^2}{n} < (2\epsilon)^2 \\ &\implies x \in B_{2\epsilon}(0), \end{aligned}$$

and the claim follows.

Observe that $\lambda^n(Q_\epsilon(0)) = \prod_{j=1}^n \frac{2\epsilon}{\sqrt{n}} > 0$. Now take some open set U. By translating it we can achieve that $0 \in U$ and, as we know, this movement does not affect $\lambda^n(U)$. As $0 \in U$ we find some $\epsilon > 0$ such that $B_\epsilon(0) \subset U$, hence

$$\lambda^n(U) \geqslant \lambda^n(B_\epsilon(0)) \geqslant \lambda(Q_\epsilon(0)) > 0.$$

(ii) For closed sets this is, *in general*, wrong. Trivial counterexample: the singleton $\{0\}$ is closed, it is Borel (take a countable sequence of nested rectangles, centered at 0 and going down to $\{0\}$) and the Lebesgue measure is zero.

To get strictly positive Lebesgue measure, one possibility is to have interior points, i.e. closed sets which have non-empty interior do have positive Lebesgue measure.

Problem 6.4 (i) Without loss of generality we can assume that $a < b$. We have $[a + \frac{1}{k}, b) \uparrow (a, b)$ as $k \to \infty$. Thus, by the continuity of measures, Theorem 4.4, we find (write $\lambda = \lambda^1$, for short)

$$\lambda(a, b) = \lim_{k\to\infty} \lambda \left[a + \frac{1}{k}, b\right) = \lim_{k\to\infty} \left(b - a - \frac{1}{k}\right) = b - a.$$

Since $\lambda[a, b) = b - a$, too, this proves again that

$$\lambda(\{a\}) = \lambda([a, b) \setminus (a, b)) = \lambda[a, b) - \lambda(a, b) = 0.$$

(ii) The hint says it all: H is contained in the union $\bigcup_{k\in\mathbb{N}} A_k$ and we have $\lambda^2(A_k) = (2\epsilon\, 2^{-k}) \cdot (2k) = 4 \cdot \epsilon \cdot k2^{-k}$. Using the σ-subadditivity and monotonicity of measures (the A_k's are clearly not disjoint) we get

$$0 \leqslant \lambda^2(H) \leqslant \lambda^2 \left(\bigcup_{k=1}^{\infty} A_k\right) \leqslant \sum_{k=1}^{\infty} \lambda(A_k) = \sum_{k=1}^{\infty} 4 \cdot \epsilon \cdot k2^{-k} = C\epsilon$$

where C is the finite (!) constant $4\sum_{k=1}^{\infty} k2^{-k}$ (check convergence!). As ϵ was arbitrary, we can let it $\to 0$ and the claim follows.

(iii) *n-dimensional version of (i):* We have $I = \bigtimes_{j=1}^{n} (a_j, b_j)$. Set $I_k := \bigtimes_{j=1}^{n} [a_j + \frac{1}{k}, b_j)$. Then $I_k \uparrow I$ as $k \to \infty$ and we have (write $\lambda = \lambda^n$, for short)

$$\lambda(I) = \lim_{k\to\infty} \lambda(I_k) = \lim_{k\to\infty} \prod_{j=1}^{n} \left(b_j - a_j - \frac{1}{k}\right) = \prod_{j=1}^{n} (b_j - a_j)\,.$$

n-dimensional version of (ii): The changes are obvious: $A_k = [-\epsilon 2^{-k}, \epsilon 2^{-k}) \times [-k, k)^{n-1}$ and $\lambda^n(A_k) = 2^n \cdot \epsilon \cdot 2^{-k} \cdot k^{n-1}$. The rest stays as before, since the sum $\sum_{k=1}^{\infty} k^{n-1}2^{-k}$ still converges to a finite value.

Problem 6.5 (i) All we have to show is that $\lambda^1(\{x\}) = 0$ for any $x \in \mathbb{R}$. But this has been shown already in problem 6.3(i).

(ii) Take the Dirac measure: δ_0. Then $\{0\}$ is an atom as $\delta_0(\{0\}) = 1$.

(iii) Let C be countable and let $\{c_1, c_2, c_3, \dots\}$ be an enumeration (could be finite, if C is finite). Since singletons are in $\mathcal{A}$, so is C as a countable union of the sets $\{c_j\}$. Using the σ-additivity of a measure we get

$$\mu(C) = \mu(\cup_{j\in\mathbb{N}}\{c_j\}) = \sum_{j\in\mathbb{N}} \mu(\{c_j\}) = \sum_{j\in\mathbb{N}} 0 = 0.$$

(iv) If $y_1, y_2, \dots, y_N$ are atoms of mass $P(\{y_j\}) \geqslant \frac{1}{k}$ we find by the additivity and monotonicity of measures

$$\begin{aligned}\frac{N}{k} &\leqslant \sum_{j=1}^{N} P(\{x_j\})\\ &= P\left(\bigcup_{j=1}^{N}\{y_j\}\right)\\ &= P(\{y_1, \dots, y_N\}) \leqslant P(\mathbb{R}) = 1\end{aligned}$$

so $\frac{N}{k} \leqslant 1$, i.e. $N \leqslant k$, and the claim in the hint (about the maximal number of atoms of given size) is shown.

Now denote, as in the hint, the atoms with measure of size $[\frac{1}{k}, \frac{1}{k-1})$ by $y_1^{(k)}, \dots y_{N(k)}^{(k)}$ where $N(k) \leqslant k$ is their number. Since

$$\bigcup_{k\in\mathbb{N}} \left[\tfrac{1}{k}, \tfrac{1}{k-1}\right) = (0, \infty)$$

we exhaust all possible sizes for atoms.

There are at most countably many (actually: finitely many) atoms in each size range. Since the number of size ranges is countable and since countably many countable sets make up a countable set, we can relabel the atoms as $x_1, x_2, x_3, \dots$ (could be finite) and, as we have seen in exercise 4.6(ii), the set-function

$$\nu := \sum_j P(\{x_j\}) \cdot \delta_{x_j}$$

(no matter whether the sum is over a finite or countably infinite set of j's) is indeed a measure on $\mathbb{R}$. But more is true: for any

Borel set A

$$\begin{aligned}\nu(A) &= \sum_j P(\{x_j\}) \cdot \delta_{x_j}(A) \\ &= \sum_{j\,:\,x_j \in A} P(\{x_j\}) \\ &= P(A \cap \{x_1, x_2, \dots\}) \leqslant P(A)\end{aligned}$$

showing that $\mu(A) := P(A) - \nu(A)$ is a positive number for each Borel set $A \in \mathcal{B}$. This means that $\mu : \mathcal{B} \to [0, \infty]$. Let us check M_1 and M_2. Using M_1, M_2 for P and ν (for them they are clear, as P, ν are measures!) we get

$$\mu(\emptyset) = P(\emptyset) - \nu(\emptyset) = 0 - 0 = 0$$

and for a disjoint sequence $(A_j)_{j \in \mathbb{N}} \subset \mathcal{B}$ we have

$$\begin{aligned}\mu\Big(\bigcup_j A_j\Big) &= P\Big(\bigcup_j A_j\Big) - \nu\Big(\bigcup_j A_j\Big) \\ &= \sum_j P(A_j) - \sum_j \nu(A_j) \\ &= \sum_j \big(P(A_j) - \nu(A_j)\big) \\ &= \sum_j \mu(A_j)\end{aligned}$$

which is M_2 for μ.

Problem 6.6 (i) Fix a sequence of numbers $\epsilon_k > 0, k \in \mathbb{N}_0$ such that $\sum_{k \in \mathbb{N}_0} \epsilon_k < \infty$. For example we could take a geometric series with general term $\epsilon_k := 2^{-k}$. Now define *open* intervals $I_k := (k - \epsilon_k, k + \epsilon_k)$, $k \in \mathbb{N}_0$ (these are open sets!) and call their union $I := \bigcup_{k \in \mathbb{N}_0} I_k$. As countable union of open sets I is again open. Using the σ-(sub-)additivity of $\lambda = \lambda^1$ we find

$$\lambda(I) = \lambda\Big(\bigcup_{k \in \mathbb{N}_0} I_k\Big) \overset{(*)}{\leqslant} \sum_{k \in \mathbb{N}_0} \lambda(I_k) = \sum_{k \in \mathbb{N}_0} 2\epsilon_k = 2 \sum_{k \in \mathbb{N}_0} \epsilon_k < \infty.$$

By 6.4(i), $\lambda(I) > 0$.

Note that in step $(*)$ equality holds (i.e. we would use σ-additivity rather than σ-subadditivity) if the I_k are pairwise disjoint. This happens, if all $\epsilon_k < \frac{1}{2}$ (think!), but to be on the safe side and in order not to have to worry about such details we use sub-additivity.

(ii) Take the open interior of the sets A_k, $k \in \mathbb{N}$, from the hint to 6.4(ii). That is, take the open rectangles $B_k := (-2^{-k}, 2^{-k}) \times (-k, k)$, $k \in \mathbb{N}$, (we choose $\epsilon = 1$ since we are after *finiteness* and not necessarily *smallness*). That these are open sets will be seen below. Now set $B = \bigcup_{k\in\mathbb{N}} B_k$ and observe that the union of open sets is always open. B is also unbounded and it is geometrically clear that B is connected as it is some kind of lozenge-shaped 'staircase' (draw a picture!) around the y-axis. Finally, by σ-subadditivity and using 6.4(ii) we get

$$\begin{aligned}\lambda^2(B) = \lambda^2\left(\bigcup_{k\in\mathbb{N}} B_k\right) &\leqslant \sum_{k\in\mathbb{N}} \lambda^2(B_k)\\ &= \sum_{k\in\mathbb{N}} 2 \cdot 2^{-k} \cdot 2 \cdot k\\ &= 4 \sum_{k\in\mathbb{N}} k \cdot 2^{-k} < \infty.\end{aligned}$$

It remains to check that an open rectangle is an open set. For this take any open rectangle $R = (a, b) \times (c, d)$ and pick $(x, y) \in R$. Then we know that $a < x < b$ and $c < y < d$ and since we have strict inequalities, we have that the smallest distance of this point to any of the four boundaries (draw a picture!) $h := \min\{|a - x|, |b - x|, |c - y|, |d - y|\} > 0$. This means that a square around (x, y) with side-length $2h$ is inside R and what we're going to do is to inscribe into this virtual square an open disk with radius h and centre (x, y). Since the circle is again in R, we are done. The equation for this disk is

$$(x', y') \in B_h(x, y) \iff (x - x')^2 + (y - y')^2 < h^2$$

Thus,

$$\begin{aligned}|x' - x| &\leqslant \sqrt{|x - x'|^2 + |y - y'|^2} < h\\ \text{and}\quad |y' - y| &\leqslant \sqrt{|x - x'|^2 + |y - y'|^2} < h\end{aligned}$$

i.e. $x - h < x' < x + h$ and $y - h < y' < y + h$ or $(x', y') \in (x - h, x + h) \times (y - h, y + h)$, which means that (x', y') is in the rectangle of sidelength $2h$ centered at (x, y). since (x', y') was an arbitrary point of $B_h(x, y)$, we are done.

(iii) No, this is impossible. Since we are in one dimension, connectedness forces us to go between points in a straight, uninterrupted line. Since the set is unbounded, this means that we must have a line of the sort (a, ∞) or $(-\infty, b)$ in our set and in both cases Lebesgue measure is infinite. In all dimensions $n > 1$, see part (ii) for two dimensions, we can, however, construct connected, unbounded open sets with finite Lebesgue measure.

Problem 6.7 Fix $\epsilon > 0$ and let $\{q_j\}_{j\in\mathbb{N}}$ be an enumeration of $\mathbb{Q} \cap [0,1]$. Then

$$U := U_\epsilon := \bigcup_{j\in\mathbb{N}} \left(q_j - \epsilon 2^{-j-1}, q_j - \epsilon 2^{-j-1}\right) \cap [0,1]$$

is a dense open set in $[0,1]$ and, because of σ-subadditivity,

$$\lambda(U) \leqslant \sum_{j\in\mathbb{N}} \lambda\left(q_j - \epsilon 2^{-j-1}, q_j - \epsilon 2^{-j-1}\right) = \sum_{j\in\mathbb{N}} \frac{\epsilon}{2^j} = \epsilon.$$

Problem 6.8 Assume first that for every $\epsilon > 0$ there is some open set $U_\epsilon \supset N$ such that $\lambda(U_\epsilon) \leqslant \epsilon$. Then

$$\lambda(N) \leqslant \lambda(U_\epsilon) \leqslant \epsilon \quad \forall \epsilon > 0,$$

which means that $\lambda(N) = 0$.

Conversely, let $\lambda^*(N) = \inf\left\{ \sum_j \lambda(U_j) \ : \ U_j \in \mathcal{O},\ \cup_{j\in\mathbb{N}} U_j \supset N\right\}$. Since for the Borel set N we have $\lambda^*(N) = \lambda(N) = 0$, the definition of the infimum guarantees that for every $\epsilon > 0$ there is a sequence of open sets $(U_j^\epsilon)_{j\in\mathbb{N}}$ covering N, i.e. such that $U^\epsilon := \bigcup_j U_j^\epsilon \supset N$. Since U^ϵ is again open we find because of σ-subadditivity

$$\lambda(N) \leqslant \lambda(U^\epsilon) = \lambda\left(\bigcup_j U_j^\epsilon\right) \leqslant \sum_j \lambda(U_j^\epsilon) \leqslant \epsilon.$$

Attention: A construction along the lines of Problem 3.12, hint to part (ii), using open sets $U^\delta := N + B_\delta(0)$ is, in general not successful:

- it is not clear that U^δ has finite Lebesgue measure (o.k. one can overcome this by considering $N \cap [-k, k]$ and then letting $k \to \infty$...)
- $U^\delta \downarrow \bar{N}$ and *not* N (unless N is closed, of course). If, say, N is a dense set of $[0,1]$, this approach leads nowhere.

Problem 6.9 Observe that the sets $C_k := \bigcup_{j=k}^{\infty} A_j$, $k \in \mathbb{N}$, decrease as $k \to \infty$—we admit less and less sets in the union, i.e. the union becomes smaller. Since P is a probability measure, $P(C_k) \leqslant 1$ and therefore Theorem 4.4(iii') applies and shows that

$$P\left(\bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} A_j\right) = P\left(\bigcap_{k=1}^{\infty} C_k\right) = \lim_{k\to\infty} P(C_k).$$

On the other hand, we can use σ-subadditivity of the measure P to get

$$P(C_k) = P\left(\bigcup_{j=k}^{\infty} A_j\right) \leqslant \textstyle\sum_{j=k}^{\infty} P(A_j)$$

but this is the tail of the convergent (!) sum $\sum_{j=1}^{\infty} P(A_j)$ and, as such, it goes to zero as $k \to \infty$. Putting these bits together, we see

$$P\left(\bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} A_j\right) = \lim_{k\to\infty} P(C_k) \leqslant \lim_{k\to\infty} \sum_{j=k}^{\infty} P(A_j) = 0,$$

and the claim follows.

Problem 6.10 (i) We can work out the 'optimal' $\mathcal{A}$-cover of (a, b):

Case 1: $a, b \in [0, 1)$. Then $[0, 1)$ is the best possible cover of (a, b), thus $\mu^*(a, b) = \mu[0, 1) = \frac{1}{2}$.

Case 2: $a, b \in [1, 2)$. Then $[1, 2)$ is the best possible cover of (a, b), thus $\mu^*(a, b) = \mu[1, 2) = \frac{1}{2}$.

Case 3: $a \in [0, 1), b \in [1, 2)$. Then $[0, 1) \cup [1, 2)$ is the best possible cover of (a, b), thus $\mu^*(a, b) = \mu[0, 1) + \mu[1, 2) = 1$.

And in the case of a singleton $\{a\}$ the best possible cover is always either $[0, 1)$ or $[1, 2)$ so that $\mu^*(\{a\}) = \frac{1}{2}$ for all a.

(ii) Assume that $(0, 1) \in \mathcal{A}^*$. Since $\mathcal{A} \subset \mathcal{A}^*$, we have $[0, 1) \in \mathcal{A}^*$, hence $\{0\} = [0, 1) \setminus (0, 1) \in \mathcal{A}^*$. Since $\mu^*(0, 1) = \mu^*(\{0\}) = \frac{1}{2}$, and since μ^* is a measure on $\mathcal{A}^*$ (cf. step 4 in the proof of Theorem 6.1), we get

$$\frac{1}{2} = \mu[0, 1) = \mu^*[0, 1) + \mu^*(0, 1) + \mu^*\{0\} = \frac{1}{2} + \frac{1}{2} = 1$$

leading to a contradiction. Thus neither $(0, 1)$ nor $\{0\}$ are elements of $\mathcal{A}^*$.

Problem 6.11 Since $\mathcal{A} \subset \mathcal{A}^*$, the only interesting sets (to which one could extend μ) are those $B \subset \mathbb{R}$ where both B and B^c are uncountable. By definition,

$$\gamma^*(B) = \inf\Big\{ \sum_j \gamma(A_j) \ : \ A_j \in \mathcal{A}, \bigcup_j A_j \supset B \Big\}.$$

The infimum is obviously attained for $A_j = \mathbb{R}$, so that $\gamma^*(B) = \gamma^*(B^c) = 1$. On the other hand, since γ^* is necessarily additive on $\mathcal{A}^*$, the assumption that $B \in \mathcal{A}^*$ leads to a contradiction:

$$1 = \gamma(\mathbb{R}) = \gamma^*(\mathbb{R}) = \gamma^*(B) + \gamma^*(B^c) = 2.$$

Thus, $\mathcal{A} = \mathcal{A}^*$.