7 Measurable mappings. Solutions to Problems 7.1–7.11

Problem 7.1 We have $\tau_x^{-1}(z) = z - x$. According to Lemma 7.2 we have to check that

$$\tau_x^{-1}([a,b)) \in \mathcal{B}^n \qquad \forall\, [a,b) \in \mathcal{J}$$

since the rectangles $\mathcal{J}$ generate $\mathcal{B}^n$. Clearly,

$$\tau_x^{-1}([a,b)) = [a,b) - x = [a-x, b-x) \in \mathcal{J} \subset \mathcal{B}^n,$$

and the claim follows.

Problem 7.2 We had $\Sigma' = \{A' \subset X' : T^{-1}(A') \in \mathcal{A}\}$ where $\mathcal{A}$ was a σ-algebra of subsets of X. Let us check the properties (Σ_1)–(Σ_3).

(Σ_1) Take $\emptyset \subset X'$. Then $T^{-1}(\emptyset) = \emptyset \in \mathcal{A}$, hence $\emptyset \in \Sigma'$.

(Σ_2) Take any $B \in \Sigma'$. Then $T^{-1}(B) \in \mathcal{A}$ and therefore $T^{-1}(B^c) = \left(T^{-1}(B)\right)^c \in \mathcal{A}$ since all set-operations interchange with inverse maps and since $\mathcal{A}$ is a σ-algebra. This shows that $B^c \in \Sigma'$.

(Σ_3) Take any sequence $(B_j)_{j\in\mathbb{N}} \subset \Sigma'$. Then, using again the fact that $\mathcal{A}$ is a σ-algebra, $T^{-1}(\cup_j B_j) = \bigcup_j T^{-1}(B_j) \in \mathcal{A}$ which proves that $\bigcup_j B_j \in \Sigma'$.

Problem 7.3 (i) First of all we remark that $T_i^{-1}(\mathcal{A}_i)$ is itself a σ-algebra, cf. Example 3.3(vii).

If $\mathcal{C}$ is a σ-algebra of subsets of X such that $T_i : (X, \mathcal{C}) \to (X_i, \mathcal{A}_i)$ becomes measurable, we know from the very definition that $T^{-1}(\mathcal{A}_i) \subset \mathcal{C}$. From this, however, it is clear that $T^{-1}(\mathcal{A}_i)$ is the minimal σ-algebra that renders T measurable.

(ii) From part (i) we know that $\sigma(T_i, i \in I)$ necessarily contains $T_i^{-1}(\mathcal{A}_i)$ for every $i \in I$. Since $\bigcup_i T_i^{-1}(\mathcal{A}_i)$ is, in general, not a σ-algebra, we have $\sigma\Big(\bigcup_i T_i^{-1}(\mathcal{A}_i)\Big) \subset \sigma(T_i, i \in I)$. On the other hand, each T_i is, because of $T_i^{-1}(\mathcal{A}_i) \subset \bigcup_i T_i^{-1}(\mathcal{A}_i) \subset \sigma(T_i, i \in I)$ measurable w.r.t. $\sigma\Big(\bigcup_i T_i^{-1}(\mathcal{A}_i)\Big)$ and this proves the claim.

Problem 7.4 We have to show that

$$f : (F, \mathcal{F}) \to (X, \sigma(T_i,\ i \in I)) \;\text{ measurable}$$

$$\iff \quad \forall\, i \in I \;:\; T_i \circ f : (F, \mathcal{F}) \to (X_i, \mathcal{A}_i) \;\text{ measurable.}$$

Now

$$\begin{aligned}
\forall\, i \in I \;:\; (T_i \circ f)^{-1}(\mathcal{A}_i) \subset \mathcal{F} &\iff \forall\, i \in I \;:\; f^{-1}\big(T_i^{-1}(\mathcal{A}_i)\big) \subset \mathcal{F} \\
&\iff f^{-1}\Big(\bigcup\nolimits_{i\in I} T_i^{-1}(\mathcal{A}_i)\Big) \subset \mathcal{F} \\
&\overset{(*)}{\iff} \sigma\Big[f^{-1}\Big(\bigcup\nolimits_{i\in I} T_i^{-1}(\mathcal{A}_i)\Big)\Big] \subset \mathcal{F} \\
&\overset{(**)}{\iff} f^{-1}\Big(\sigma\Big[\bigcup\nolimits_{i\in I} T_i^{-1}(\mathcal{A}_i)\Big]\Big) \subset \mathcal{F}.
\end{aligned}$$

Only (*) and (**) are not immediately clear. The direction '$\Longleftarrow$' in (*) is trivial, while '$\Longrightarrow$' follows if we observe that the right-hand side, $\mathcal{F}$, is a σ-algebra. The equivalence (**) is another case of Problem 7.8 (see there for the solution!).

Problem 7.5 Using the notation of the foregoing Problem 7.4 we put $I = \{1, 2, \dots, m\}$, $T_j := \pi_j : \mathbb{R}^m \to \mathbb{R}$, $\pi_j(x_1, \dots, x_m) := x_j$ is the coordinate projection, $\mathcal{A}_j := \mathcal{B}(\mathbb{R})$. Since each π_j is continuous, we have $\sigma(\pi_1, \dots, \pi_m) \subset \mathcal{B}(\mathbb{R}^m)$ so that Problem 7.4 applies and proves

$$\begin{gathered}
f \;\text{ is } \mathcal{B}(\mathbb{R}^m)\text{-measurable} \iff \\
f_j = \pi_j \circ f \;\text{ is } \mathcal{B}(\mathbb{R})\text{-measurable for all } j = 1, 2, \dots, m.
\end{gathered}$$

Remark. We will see, in fact, in Chapter 13 (in particular in Theorem 13.10) that we have the equality $\sigma(\pi_1, \dots, \pi_m) = \mathcal{B}(\mathbb{R}^m)$.

Problem 7.6 In general the direct image $T(\mathcal{A})$ of a σ-algebra is not any longer a σ-algebra. (Σ_1) and (Σ_3) hold, but (Σ_2) will, in general, fail. Here is an example: Take $X = X' = \mathbb{N}$, take any σ-algebra $\mathcal{A}$ other than $\{\emptyset, \mathbb{N}\}$ in $\mathbb{N}$, and let $T : \mathbb{N} \to \mathbb{N}$, $T(j) = 1$ be the constant map. Then $T(\emptyset) = \emptyset$ but $T(A) = \{1\}$ whenever $A \neq \emptyset$. Thus, $\{1\} = T(A^c) \neq [T(A)]^c = \mathbb{N} \setminus \{1\}$ but equality would be needed if $T(\mathcal{A})$ were a σ-algebra. This means that Σ_2 fails.

Necessary and sufficient for $T(\mathcal{A})$ to be a σ-algebra is, clearly, that T^{-1} is a measurable map $T^{-1} : X' \to X$.

Problem 7.7 Consider for $t > 0$ the dilation $m_t : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto t \cdot x$. Since m_t is continuous, it is Borel measurable. Moreover, $m_t^{-1} = m_{1/t}$ and so

$$t \cdot B = m_{1/t}^{-1}(B)$$

which shows that $\lambda^n(t \cdot B) = \lambda^n \circ m_{1/t}^{-1}(B) = m_{1/t}(\lambda^n)(B)$ is actually an image measure of λ^n. Now show the formula first for rectangles $B = \times_{j=1}^{n} [a_j, b_j)$ (as in Problem 5.8) and deduce the statement from the uniqueness theorem for measures.

Problem 7.8 We have

$$T^{-1}(\mathcal{G}) \subset \underbrace{T^{-1}(\sigma(\mathcal{G}))}_{\text{is itself a } \sigma\text{-algebra}} \implies \sigma(T^{-1}(\mathcal{G})) \subset T^{-1}(\sigma(\mathcal{G})).$$

For the converse consider $T : (X, \sigma(T^{-1}(\mathcal{G}))) \to (Y, \sigma(\mathcal{G}))$. By the very choice of the σ-algebras and since $T^{-1}(\mathcal{G}) \subset \sigma(T^{-1}(\mathcal{G}))$ we find that T is $\sigma(T^{-1}(\mathcal{G}))/\sigma(\mathcal{G})$ measurable—mind that we only have to check measurability at a generator (here: $\mathcal{G}$) in the image region. Thus,

$$T^{-1}(\sigma(\mathcal{G})) \subset \sigma(T^{-1}(\mathcal{G})).$$

Problem 7.9 (i) **Note the misprint:** we need to assume that $\mu[-n, n) < \infty$ for all $n \in \mathbb{N}$.

Monotonicity: If $x \leqslant 0 \leqslant y$, then $F_\mu(x) \leqslant 0 \leqslant F_\mu(y)$.

If $0 < x \leqslant y$, we have $[0, x) \subset [0, y)$ and so $0 \leqslant F_\mu(x) = \mu[0, x) \leqslant \mu[0, y) = F_\mu(y)$.

If $x \leqslant y < 0$, we have $[y, 0) \subset [x, 0)$ and so $0 \leqslant -F_\mu(y) = \mu[y, 0) \leqslant \mu[x, 0) = -F_\mu(x)$, i.e. $F_\mu(x) \leqslant F_\mu(y) \leqslant 0$.

Left-continuity: Let us deal with the case $x \geqslant 0$ only, the case $x < 0$ is analogous (and even easier). Assume first that $x > 0$. Take any sequence $x_k < x$ and $x_k \uparrow x$ as $k \to \infty$. Without loss of generality we can assume that $0 < x_k < x$. Then $[0, x_k) \uparrow [0, x)$ and using Theorem 4.4(iii') implies

$$\lim_{k\to\infty} F_\mu(x_k) = \lim_{k\to\infty} \mu[0, x_k) = \mu[0, x) = F_\mu(x).$$

If $x = 0$ we must take a sequence $x_k < 0$ and we have then $[x_k, 0) \downarrow [0, 0) = \emptyset$. Again by Theorem 4.4, now (iii”), we get

$$\lim_{k\to\infty} F_\mu(x_k) = -\lim_{k\to\infty} \mu[x_k, 0) = \mu(\emptyset) = 0 = F_\mu(0).$$

which shows left-continuity at this point, too.

We remark that, since for a sequence $y_k \downarrow y$, $y_k > y$ we have $[0, y_k) \downarrow [0, y]$, and not $[0, y)$, we cannot expect right-continuity in general.

(ii) Since $\mathcal{J} = \{[a,b), a \leqslant b\}$ is a semi-ring (cf. Proposition 6.4) it is enough to check that ν_F is a premeasure on $\mathcal{J}$. This again amounts to showing (M_1) and (M_2) relative to $\mathcal{J}$ (mind you: ν_F is not a *measure* as $\mathcal{J}$ is not a σ-algebra....). We do this in the equivalent form of Theorem 4.4, i.e. we prove (i), (ii) and (iii') of Theorem 4.4:

(i) $\nu_F(\emptyset) = \nu_F[a,a) = F(a) - F(a) = 0$ for any a.

(ii) Let $a \leqslant b \leqslant c$ so that $[a,b), [b,c) \in \mathcal{J}$ are disjoint sets and $[a,c) = [a,b) \cup [b,c) \in \mathcal{J}$ (the latter is crucial). Then we have

$$\begin{aligned}\nu_F[a,b) + \nu_F[b,c) &= F(b) - F(a) + F(c) - F(b)\\ &= F(c) - F(a)\\ &= \nu_F[a,c)\\ &= \nu_F\big([a,b) \cup [b,c)\big).\end{aligned}$$

(iii') (Sufficient since ν_F is finite for every set $[a,b)$). Now take a sequence of intervals $[a_k, b_k)$ which decreases towards some $[a,b) \in \mathcal{J}$. This means that $a_k \uparrow a$, $a_k \leqslant a$ and $b_k \downarrow b$, $b_k \geqslant b$ because the intervals are nested (gives increasing-decreasing sequences). If $b_k > b$ for infinitely many k, this would mean that $[a_k, b_k) \to [a,b] \notin \mathcal{J}$ since $b \in [a_k, b_k)$ for all k. Since we are only interested in sequences whose limits stay in $\mathcal{J}$, the sequence b_k must reach b after finitely many steps and stay there to give $[a,b)$. Thus, we may assume directly that we have only $[a_k, b) \downarrow [a,b)$ with $a_k \uparrow a$, $a_k \leqslant a$. But then we can use left-continuity and get

$$\begin{aligned}\lim_{k\to\infty} \nu_F[a_k, b) = \lim_{k\to\infty} (F(b) - F(a_k)) &= F(b) - F(a)\\ &= \nu_F[a,b).\end{aligned}$$

Note that ν_F takes on only positive values because F increases.

This means that we find *at least one* extension. Uniqueness follows since $\nu_F[-k,k) = F(k) - F(-k) < \infty$ and $[-k,k) \uparrow \mathbb{R}$.

(iii) Now let μ be a measure with $\mu[-n,n) < \infty$. The latter means that the function $F_\mu(x)$, as defined in part (i), is finite for every $x \in \mathbb{R}$. Now take this F_μ and define, as in (ii) a (uniquely defined) measure ν_{F_μ}. Let us see that $\mu = \nu_{F_\mu}$. For this, it is enough to show equality on the sets of type $[a,b)$ (since such sets generate the Borel sets and the uniqueness theorem applies....)

If $0 \leqslant a \leqslant b$,

$$\begin{aligned}\nu_{F_\mu}[a,b) = F_\mu(b) - F_\mu(a) &= \mu[0,b) - \mu[0,a) \\ &= \mu\big([0,b) \setminus [0,a)\big) \\ &= \mu[a,b) \quad \checkmark\end{aligned}$$

If $a \leqslant b \leqslant 0$,

$$\begin{aligned}\nu_{F_\mu}[a,b) = F_\mu(b) - F_\mu(a) &= -\mu[b,0) - (-\mu[a,0)) \\ &= \mu[a,0) - \mu[b,0) \\ &= \mu\big([a,0) \setminus [b,0)\big) \\ &= \mu[a,b) \quad \checkmark\end{aligned}$$

If $a \leqslant 0 \leqslant b$,

$$\begin{aligned}\nu_{F_\mu}[a,b) = F_\mu(b) - F_\mu(a) &= \mu[0,b)) - (-\mu[a,0)) \\ &= \mu[a,0)) + \mu[0,b) \\ &= \mu\big([a,0) \mathbin{\dot\cup} [0,b)\big) \\ &= \mu[a,b) \quad \checkmark\end{aligned}$$

(iv) $F : \mathbb{R} \to \mathbb{R}$ with $F(x) = x$, since $\lambda[a,b) = b - a = F(b) - F(a)$.

(v) $F : \mathbb{R} \to \mathbb{R}$, with, say, $F(x) = \begin{cases} 0, & x \leqslant 0 \\ 1, & x > 0 \end{cases} = \mathbf{1}_{(0,\infty)}(x)$ since $\delta_0[a,b) = 0$ whenever $a, b < 0$ or $a, b > 0$. This means that F must be constant on $(-\infty, 0)$ and $(0, \infty)$ If $a \leqslant 0 < b$ we have, however, $\delta_0[a,b) = 1$ which indicates that $F(x)$ must jump by 1 at the point 0. Given the fact that F must be left-continuous, it is clear that it has, in principle, the above form. The only ambiguity is, that if $F(x)$ does the job, so does $c + F(x)$ for any constant $c \in \mathbb{R}$.

(vi) Assume that F is continuous at the point x. Then

$$\begin{aligned}\mu(\{x\}) &= \mu\Big(\bigcap_{k\in\mathbb{N}} \big[x, x + \tfrac{1}{k}\big)\Big) \\ &\overset{4.4}{=} \lim_{k\to\infty} \mu\left(\big[x, x + \tfrac{1}{k}\big)\right) \\ &\overset{\text{def}}{=} \lim_{k\to\infty} \left(F\left(x + \tfrac{1}{k}\right) - F(x)\right) \\ &= \lim_{k\to\infty} F\left(x + \tfrac{1}{k}\right) - F(x)\end{aligned}$$

$$\overset{(*)}{=} F(x) - F(x) = 0$$

where we used (right-)continuity of F at x in the step marked $(*)$. Now, let conversely $\mu(\{x\}) = 0$. A similar calculation as above shows, that for *every* sequence $\epsilon_k > 0$ with $\epsilon_k \to \infty$

$$\begin{aligned} F(x+) - F(x) &= \lim_{k\to\infty} F(x + \epsilon_k) - F(x) \\ &\overset{\text{def}}{=} \lim_{k\to\infty} \mu[x, x + \epsilon_k) \\ &\overset{4.4}{=} \mu\Big(\bigcap_{k\in\mathbb{N}} [x, x + \epsilon_k)\Big) \\ &= \mu(\{x\}) = 0 \end{aligned}$$

which means that $F(x) = F(x+)$ ($x+$ indicates the right limit), i.e. F is right-continuous at x, hence continuous, as F is left-continuous anyway.

(vii) Then hint is indeed already the proof. Almost, that is... Let μ be some measure as specified in the problem. From part (iii) we know that the Stieltjes function $F := F_\mu$ then satisfies

$$\begin{aligned} \mu[a, b) = F(b) - F(a) &= \lambda^1[F(a), F(b)) \\ &\overset{(\#)}{=} \lambda^1(F([a, b))) \\ &\overset{(\#\#)}{=} \lambda^1 \circ F([a, b)). \end{aligned}$$

The crunching points in this argument are the steps (#) and (##).

(#) This is o.k. since F was continuous, and the intermediate value theorem for continuous functions tells us that intervals are mapped to intervals. So, no problem here, just a little thinking needed.

(##) This is more subtle. We have defined image measures *only* for inverse maps, i.e. for expressions of the type $\lambda^1 \circ G^{-1}$ where G was measurable. So our job is to see that F can be obtained in the form $F = G^{-1}$ where G is measurable. In other words, we have to invert F. The problem is that we need to understand that, if $F(x)$ is flat on some interval (a, b) inversion becomes a problem (since then F^{-1} has a jump—horizontals become verticals in inversions, as inverting is somehow the mirror-image w.r.t. the 45-degree line in the coordinate system.).

So, if there are no flat bits, then this means that F is strictly increasing, and it is clear that G exists and is even continuous there.

If we have a flat bit, let's say exactly if $x \in [a,b]$ and call $F(x) = F(a) = F(b) = C$ for those x; clearly, F^{-1} jumps at C and we must see to it that we take a version of F^{-1}, say one which makes F^{-1} left-continuous at C—note that we could assign any value from $[a,b]$ to $F^{-1}(C)$—which is accomplished by setting $F^{-1}(C) = a$. (Draw a graph to illustrate this!)

There is A canonical expression for such a 'generalized' left-continuous inverse of an increasing function (which may have jumps and flat bits—jumps of F become just flat bits in the graph of F^{-1}, think!) and this is:

$$G(y) = \inf\{x \;:\; F(x) \geqslant y\}$$

Let us check measurability:

$$\begin{aligned} y_0 \in \{G \geqslant \lambda\} &\iff G(y_0) \geqslant \lambda \\ &\overset{\text{def}}{\iff} \inf\{F \geqslant y_0\} \geqslant \lambda \\ &\overset{(\ddagger)}{\implies} F(\lambda) \leqslant y_0 \\ &\iff y_0 \in [F(\lambda), \infty). \end{aligned}$$

Since F is monotonically increasing, we find also '$\Longleftarrow$' in step ($\ddagger$), hence

$$\{G \geqslant \lambda\} = [F(\lambda), \infty) \in \mathcal{B}(\mathbb{R})$$

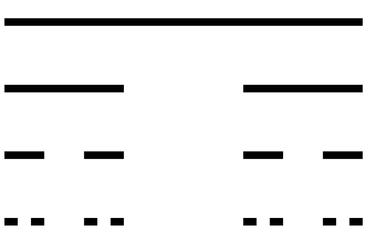
which shows that G is measurable. Even more: it shows that $G^{-1}(x) := \inf\{G \geqslant \lambda\} = F(x)$. Thus, $\lambda^1 \circ F = \lambda^1 \circ G^{-1} = \mu$ is indeed an image measure of λ^1.

(viii) We have $F(x) = F_{\delta_0}(x) = \mathbf{1}_{(0,\infty)}(x)$ and its left-continuous inverse $G(y)$ in the sense of part (vii) is given by

$$G(y) = \begin{cases} +\infty, & y > 1 \\ 0, & 0 < y \leqslant 1 \\ -\infty, & y \leqslant 0 \end{cases}.$$

This function is clearly measurable (use $\bar{\mathcal{B}}$ to accommodate $\pm\infty$) and so the claim holds in this case. Observe that in this case F is not any longer continuous but only left-continuous.

Problem 7.10 (i) We find the following picture:



$E_0 = [0,1]$

$E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

$E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$

$E_3 = [0, \frac{1}{27}] \cup [\frac{2}{27}, \frac{1}{9}] \cup [\frac{2}{9}, \frac{7}{27}] \cup [\frac{8}{27}, \frac{1}{3}] \cup$
$[\frac{2}{3}, \frac{19}{27}] \cup [\frac{20}{27}, \frac{7}{9}] \cup [\frac{8}{9}, \frac{25}{27}] \cup [\frac{26}{27}, 1]$

(ii) Each E_n is a finite union of 2^n closed and bounded intervals. As such, E_n is itself a closed and bounded set, hence compact. The intersection of closed and bounded sets is again closed and bounded, so compact. This shows that C is compact. That C is non-empty follows from the intersection principle: if one has a nested sequence of non-empty compact sets, their intersection is not empty. (This is sometimes formulated in a somewhat stronger form and called: *finite intersection property.* The general version is then: Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of compact sets such that *each finite* sub-family has non-void intersection, then $\bigcap_n K_n \neq \emptyset$). This is an obvious generalization of the interval principle: nested non-void closed and bounded intervals have a non-void intersection.

(iii) At step n we remove open middle-third intervals of length 3^{-n}. To be precise, we partition E_{n-1} in pieces of length 3^{-n} and remove every other interval. The same effect is obtained if we partition $[0,\infty)$ in pieces of length 3^{-n} and remove every other piece. Call the taken out pieces F_n and set $E_n = E_{n-1} \setminus F_n$, i.e. we remove from E_{n-1} even pieces which were already removed in previous steps. It is clear that F_n exactly consists of sets of the form $(\frac{3k+1}{3^n}, \frac{3k+2}{3^n})$, $k \in \mathbb{N}_0$ which comprises exactly 'every other' set of length 3^{-n}. Since we do this for every n, the set C is disjoint to the union of these intervals over $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$.

(iv) Since E_n consists of 2^n intervals $I_1 \uplus \ldots \uplus I_{2^n}$, each of which has length 3^{-n} (prove this by a trivial induction argument!), we get

$$\lambda(E_n) = \lambda(I_1) + \ldots + \lambda(I_{2^n}) = 2^n \cdot 3^{-n} = \left(\frac{2}{3}\right)^n$$

where we also used (somewhat pedantically) that

$$\lambda[a,b] = \lambda([a,b) \uplus \{b\}) = \lambda[a,b) + \lambda\{b\} = b - a + 0 = b - a.$$

Now using Theorem 4.4 we conclude that $\lambda(C) = \inf_n \lambda(E_n) = 0$.

(v) Fix $\epsilon > 0$ and choose n so big that $3^{-n} < \epsilon$. Then E_n consists of 2^n *disjoint* intervals of length $3^{-n} < \epsilon$ and cannot possibly contain a ball of radius ϵ. Since $C \subset E_n$, the same applies to C. Since ϵ was arbitrary, we are done. (Remark: an open ball in $\mathbb{R}$ with centre x is obviously an open interval with midpoint x, i.e. $(x-\epsilon, x+\epsilon)$.)

(vi) Fix n and let $k = 0, 1, 2, \ldots, 3^{n-1}-1$. We saw in (c) that at step n we remove the intervals F_n, i.e. the intervals of the form

$$\left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right) = \Big(0.\underbrace{***\ldots*1}_{n}000\ldots,\ 0.\underbrace{***\ldots*2}_{n}000\ldots\Big)$$

where we used the ternary representation of x. These are exactly the numbers in $[0,1]$ whose ternary expansion has a 1 at the nth digit. As $0.***\ldots*1 = 0.***\ldots*022222\ldots$ has two representations, the left endpoint stays in. Since we do this for every step $n \in \mathbb{N}$, the claim follows.

(vii) Take $t \in C$ with ternary representation $t = 0.t_1t_2t_3\ldots t_j\ldots$, $t_j \in \{0,2\}$ and map it to the binary number $b = 0.\frac{t_1}{2}\frac{t_2}{2}\frac{t_3}{2}\ldots\frac{t_j}{2}$ with digits $b_j = \frac{t_j}{2} \in \{0,1\}$. This gives a bijection between C and $[0,1]$, i.e. both have 'as infinitely many' points, i.e. $\#C = \#[0,1]$. Despite of that

$$\lambda(C) = 0 \neq 1 = \lambda([0,1])$$

which is, by the way, another proof for the fact that σ-additivity for the Lebesgue measure does not extend to general uncountable unions.

Problem 7.11 One direction is easy: if $f = g \circ T$ with $g : Y \to \mathbb{R}$ being measurable, we have

$$f^{-1}(\mathcal{B}(\mathbb{R})) = (g\circ T)^{-1}(\mathcal{B}(\mathbb{R})) = T^{-1}(g^{-1}(\mathcal{B}(\mathbb{R}))) \subset T^{-1}(\mathcal{A}) = \sigma(T).$$

Conversely, if f is $\sigma(T)$-measurable, then whenever $T(x) = T(x')$, we have $f(x) = f(x')$; for if not, let B be a Borel set in $\mathbb{R}$ with $f(x) \in B$ and $f(x') \notin B$. Then $f^{-1}(B) = T^{-1}(C)$ for some $C \in \mathcal{A}$, with $T(x) \in C$ but $T(x') \notin C$—which is impossible. Thus, $f = g \circ T$ for some function g from the range $T(X)$ of T. But, by assumption, $T(X) = Y$.

For any Borel set $S \subset \mathbb{R}$, $T^{-1} \circ g^{-1}(S) = f^{-1}(S) = T^{-1}(A)$ for some suitable $A \in \mathcal{A}$, so $A = g^{-1}(S)$ proving the measurability of g.

Remark. Originally, I had in mind the above solution (taken from Dudley's book [14], Theorem 4.2.8), but recently I found a much simpler solution (below) which makes the whole Remark following the statement of Problem 7.11 obsolete; moreover, it is not any longer needed to have T surjective. However, this requires that you read through Theorem 8.8 from the next chapter.

Alternative solution: One direction is easy: if $f = g \circ T$ with $g : Y \to \mathbb{R}$ being measurable, we have

$$f^{-1}(\mathcal{B}(\mathbb{R})) = (g \circ T)^{-1}(\mathcal{B}(\mathbb{R})) = T^{-1}(g^{-1}(\mathcal{B}(\mathbb{R}))) \subset T^{-1}(\mathcal{A}) = \sigma(T).$$

For the converse assume in ...

Step 1: ...first that $f = \mathbf{1}_B$ is a step function consisting of a single step. Then

$$\begin{aligned} \mathbf{1}_B \quad \sigma(T)\text{-measurable} &\iff B \in \sigma(T) \\ &\iff \exists\, A \in \mathcal{A} \,:\, B = T \in A = T^{-1}(A) \\ &\iff \mathbf{1}_B = \mathbf{1}_A \circ T \end{aligned}$$

so that $g = \mathbf{1}_A : Y \to \mathbb{R}$ does the job.

Step 2: If $f = \sum_{j=1}^N \alpha_j \mathbf{1}_{B_j}$ then, by Step 1, $g = \sum_{j=1}^N \alpha_j \mathbf{1}_{A_j}$ with the obvious notation $B_j = \{T \in A_j\}$, $A_j \in \mathcal{A}$ suitable, and $f = g \circ T$ and $g : Y \to \mathbb{R}$.

Step 3: If $f \geqslant 0$ is measurable, then we use Theorem 8.8 which says that we can approximate f as an increasing limit of $\sigma(T)$-measurable (!) elementary functions (have a look at the proof of 8.8!), say $f = \sup_j f_j$ and each f_j is of the form of the functions from Step 2. Thus, there are again functions g_j such that $f_j = g_j \circ T$ and we get

$$f = \limsup_j f_j = \limsup_j g_j \circ T = (\limsup_j g_j) \circ T$$

which means that $g := \limsup_j g_j : Y \to \mathbb{R}$ does the job.

Step 4: If f is just measurable, consider positive and negative parts $f = f^+ - f^-$ and construct, according to Step 3, $g^\pm : Y \to \mathbb{R}$ such that $g^\pm \circ T = f^\pm$. Then $g := g^+ - g^-$ does the trick.

8 Measurable functions. Solutions to Problems 8.1–8.18

Problem 8.1 We remark, first of all, that $\{u \geqslant \alpha\} = u^{-1}([x,\infty))$ and, similarly, for the other sets. Now assume that $\{u \geqslant \beta\} \in \mathcal{A}$ for all β. Then

$$\begin{aligned}
\{u > \alpha\} = u^{-1}((\alpha,\infty)) &= u^{-1}\Bigg(\bigcup_{k\in\mathbb{N}} \left[\alpha + \tfrac{1}{k}, \infty\right)\Bigg) \\
&= \bigcup_{k\in\mathbb{N}} u^{-1}\left(\left[\alpha + \tfrac{1}{k}, \infty\right)\right) \\
&= \bigcup_{k\in\mathbb{N}} \underbrace{\left\{u \geqslant \alpha + \tfrac{1}{k}\right\}}_{\text{by assumption } \in \mathcal{A}} \in \mathcal{A}
\end{aligned}$$

since $\mathcal{A}$ is a σ-algebra.

Conversely, assume that $\{u > \beta\} \in \mathcal{A}$ for all β. Then

$$\begin{aligned}
\{u \geqslant \alpha\} = u^{-1}([\alpha,\infty)) &= u^{-1}\Bigg(\bigcap_{k\in\mathbb{N}} \left(\alpha - \tfrac{1}{k}, \infty\right)\Bigg) \\
&= \bigcap_{k\in\mathbb{N}} u^{-1}\left(\left(\alpha - \tfrac{1}{k}, \infty\right)\right) \\
&= \bigcap_{k\in\mathbb{N}} \underbrace{\left\{u > \alpha - \tfrac{1}{k}\right\}}_{\text{by assumption } \in \mathcal{A}} \in \mathcal{A}.
\end{aligned}$$

since $\mathcal{A}$ is a σ-algebra. Finally, as

$$\{u > \alpha\}^c = \{u \leqslant \alpha\} \qquad \text{and} \qquad \{u \geqslant \alpha\}^c = \{u < \alpha\}$$

we have that $\{u > \alpha\} \in \mathcal{A}$ if, and only if, $\{u \leqslant \alpha\} \in \mathcal{A}$ and the same holds for the sets $\{u \geqslant \alpha\}, \{u < \alpha\}$.

Problem 8.2 Recall that $B^* \in \bar{\mathcal{B}}$ if, and only if $B^* = B \cup C$ where $B \in \mathcal{B}$ and C is any of the following sets: $\emptyset, \{-\infty\}, \{\infty\}, \{-\infty, \infty\}$. Using the fact that $\mathcal{B}$ is a σ-algebra and using this notation (that is: $\bar{\mathcal{B}}$-sets carry an asterisk *) we see

(Σ_1) Take $B = \emptyset \in \mathcal{B}$, $C = \emptyset$ to see that $\emptyset^* = \emptyset \cup \emptyset \in \bar{\mathcal{B}}$;

(Σ_2) Let $B^* \in \overline{\mathcal{B}}$. Then (complements are to be taken in $\bar{\mathcal{B}}$

$$\begin{aligned}(B^*)^c &= (B \cup C)^c \\ &= B^c \cap C^c \\ &= (\overline{\mathbb{R}} \setminus B) \cap (\overline{\mathbb{R}} \setminus C) \\ &= (\mathbb{R} \setminus B \cup \{-\infty, +\infty\}) \cap (\overline{\mathbb{R}} \setminus C) \\ &= ((\mathbb{R} \setminus B) \cap (\overline{\mathbb{R}} \setminus C)) \cup (\{-\infty, +\infty\} \cap (\overline{\mathbb{R}} \setminus C)) \\ &= (\mathbb{R} \setminus B) \cup (\{-\infty, +\infty\} \cap (\overline{\mathbb{R}} \setminus C))\end{aligned}$$

which is again of the type $\mathcal{B}$-set union a set of the list $\emptyset, \{-\infty\}$, $\{\infty\}$, $\{-\infty, \infty\}$, hence it is in $\bar{\mathcal{B}}$.

(Σ_3) Let $B_n^* \in \overline{\mathcal{B}}$ and $B_n^* = B_n \cup C_n$. Then

$$B^* = \bigcup_{n\in\mathbb{N}} B_n^* = \bigcup_{n\in\mathbb{N}} (B_n \cup C_n) = \bigcup_{n\in\mathbb{N}} B_n \cup \bigcup_{n\in\mathbb{N}} C_n = B \cup C$$

with $B \in \mathcal{B}$ and C from the list $\emptyset, \{-\infty\}, \{\infty\}, \{-\infty, \infty\}$, hence $B^* \in \bar{\mathcal{B}}$.

A problem is the notation $\bar{\mathcal{B}} = \mathcal{B}(\bar{\mathbb{R}})$. While the left-hand side can easily be defined by (8.5), $\mathcal{B}(\bar{\mathbb{R}})$ has a well-defined meaning as the (topological) Borel σ-algebra over the set $\bar{\mathbb{R}}$, i.e. the σ-algebra in $\bar{\mathbb{R}}$ which is defined via the open sets in $\bar{\mathbb{R}}$. To describe the open sets $(\bar{\mathbb{R}})$ of $\bar{\mathbb{R}}$ we use require, that each point $x \in U^* \in \mathcal{O}(\bar{\mathbb{R}})$ admits an open neighbourhood $B(x)$ inside U^*. If $x \neq \pm\infty$, we take $B(x)$ as the usual open ϵ-interval around x with $\epsilon > 0$ sufficiently small. If $x = \pm\infty$ we take half-lines $[-\infty, a)$ or $(b, +\infty]$ respectively with $|a|, |b|$ sufficiently large. Thus, $\mathcal{O}(\bar{\mathbb{R}})$ adds to $\mathcal{O}(\mathbb{R})$ a few extra sets and open sets are therefore of the form $U^* = U \cup C$ with $U \in \mathcal{O}(\mathbb{R})$ and C being of the form $[-\infty, a)$ or $(b, +\infty]$ or $\emptyset$ or $\bar{\mathbb{R}}$ or unions thereof.

Thus, $\mathcal{O}(\mathbb{R}) = \mathbb{R} \cap \mathcal{O}(\bar{\mathbb{R}})$ and therefore

$$\mathcal{B}(\mathbb{R}) = \mathbb{R} \cap \mathcal{B}(\bar{\mathbb{R}})$$

(this time in the proper topological sense).

Problem 8.3 (i) Notice that the indicator functions $\mathbf{1}_A$ and $\mathbf{1}_{A^c}$ are measurable. By Corollary 8.10 sums and products of measurable functions are again measurable. Since $h(x)$ can be written in the form $h(x) = \mathbf{1}_A(x)f(x) + \mathbf{1}_{A^c}(x)g(x)$, the claim follows.

(ii) The condition $f_j|_{A_j \cap A_k} = f_k|_{A_j \cap A_k}$ just guarantees that $f(x)$ is well-defined if we set $f(x) = f_j(x)$ for $x \in A_j$. Using $\bigcup_j A_j = X$ we find for $B \in \mathcal{B}(\mathbb{R})$

$$f^{-1}(B) = \bigcup_{j \in \mathbb{N}} A_j \cap f^{-1}(B) = \bigcup_{j \in \mathbb{N}} \underbrace{A_j \cap f_j^{-1}(B)}_{\in \mathcal{A}} \in \mathcal{A}.$$

An **alternative solution** would be to make the A_j's disjoint, e.g. by setting $C_1 := A_1$, $C_k := A_k \setminus (A_1 \cup \dots \cup A_{k-1})$. Then

$$f = \sum_j \mathbf{1}_{C_j} f = \sum_j \mathbf{1}_{C_j} f_j$$

and the claim follows from Corollaries 8.10 and 8.9.

Problem 8.4 Since $\mathbf{1}_B$ is $\mathcal{B}$-measurable if, and only if, $B \in \mathcal{B}$ the claim follows by taking $B \in \mathcal{B}$ such that $B \notin \mathcal{A}$ (this is possible as $\mathcal{B} \subsetneq \mathcal{A}$.

Problem 8.5 By definition, $f \in \mathcal{E}$ if it is a step-function of the form $f = \sum_{j=0}^N a_j \mathbf{1}_{A_j}$ with some $a_j \in \mathbb{R}$ and $A_j \in \mathcal{A}$. Since

$$f^+ = \sum_{\substack{0 \leqslant j \leqslant N \\ a_j \geqslant 0}} a_j \mathbf{1}_{A_j} \quad \text{and} \quad f^- = \sum_{\substack{0 \leqslant j \leqslant N \\ a_j \leqslant 0}} a_j \mathbf{1}_{A_j},$$

$f^\pm$ are again of this form and therefore simple functions.

The converse is also true since $f_f^+ - f^-$—see (8.9) or Problem 8.6—and since sums and differences of simple functions are again simple.

Problem 8.6 By definition

$$u^+(x) = \max\{u(x), 0\} \quad \text{and} \quad u^-(x) = -\min\{u(x), 0\}.$$

Now the claim follows from the elementary identities that for any two numbers $a, b \in \mathbb{R}$

$$a + 0 = \max\{a, 0\} + \min\{a, 0\} \quad \text{and} \quad |a| = \max\{a, 0\} - \min\{a, 0\}$$

which are easily verified by considering all possible cases $a \leqslant 0$ resp. $a \geqslant 0$.

Problem 8.7 Assume that $0 \leqslant u(x) \leqslant c$ for all x and some constant c. Choose $j \in \mathbb{N}$ such that $j > c$. Then the procedure used to approximate u in the proof of Theorem 8.8—see page 62, line 9 from above—guarantees that $|f_j(x) - u(x)| \leqslant 2^{-j}$ for all values of x; note that the case $u \geqslant j$ does not occur! This means that $\sup|f_j - u^+| \leqslant 2^{-j}$, i.e. we have uniform convergence.

The general case is now obtained by considering positive and negative parts $u = u^+ - u^-$ which are bounded since $u^\pm \leqslant |u| \leqslant c$.

Problem 8.8 If we show that $\{u > \alpha\}$ is an open set, it is also a Borel set, hence u is measurable.

Let us first understand what openness means: $\{u > \alpha\}$ is open means that for $x \in \{u > \alpha\}$ we find some (symmetric) neighbourhood (a 'ball') of the type $(x-h, x+h) \subset \{u > \alpha\}$. What does this mean? Obviously, that $u(y) > \alpha$ for any $y \in (x-h, x+h)$ and, in other words, $u(y) > \alpha$ whenever y is such that $|x-y| < h$. And this is the hint of how to use continuity: we use it in order to find the value of h.

u being continuous at x means that

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall y \,:\, |x-y| < \delta \,:\, |u(x) - u(y)| < \epsilon.$$

Since $u(x) > \alpha$ we know that for a sufficiently small ϵ we still have $u(x) \geqslant \alpha + \epsilon$. Take this ϵ and find the corresponding δ. Then

$$u(x) - u(y) \leqslant |u(x) - u(y)| < \epsilon \qquad \forall\, |x-y| < \delta$$

and since $\alpha + \epsilon \leqslant u(x)$ we get

$$\alpha + \epsilon - u(y) < \epsilon \qquad \forall\, |x-y| < \delta$$

i.e. $u(y) > \alpha$ for y such that $|x-y| < \delta$. This means, however, that $h = \delta$ does the job.

Problem 8.9 The minimum/maximum of two numbers $a, b \in \mathbb{R}$ can be written in the form

$$\min\{a,b\} = \frac{1}{2}\big(a + b - |a-b|\big)$$
$$\max\{a,b\} = \frac{1}{2}\big(a + b + |a-b|\big)$$

which shows that we can write $\min\{x, 0\}$ and $\max\{x, 0\}$ as a combination of continuous functions. As such they are again continuous, hence measurable. Thus,

$$u^+ = \max\{u, 0\}, \qquad u^- = -\min\{u, 0\}$$

are compositions of measurable functions, hence measurable.

Problem 8.10 The f_j are step-functions where the bases of the steps are the sets A_k^j and A_j. Since they are of the form, e.g. $\{k2^{-j} \leqslant u < (k+1)2^{-j}\} = \{k2^{-j} \leqslant u\} \cap \{u < (k+1)2^{-j}\}$, it is clear that they are not only in $\mathcal{A}$ but in $\sigma(u)$.

Problem 8.11 Corollary 8.11 If $u^\pm$ are measurable, it is clear that $u = u^+ - u^-$ is measurable since differences of measurable functions are measurable.

(For the converse we could use the previous Problem 8.10, but we give an alternative proof...) Conversely, let u be measurable. Then $s_n \uparrow u$ (this is short for: $\lim_{n\to\infty} s_n(x) = u(x)$ and this is an increasing limit) for some sequence of simple functions s_n. Now it is clear that $s_n^+ \uparrow u^+$, and s_n^+ is simple, i.e. u^+ is measurable. As $u = u^+ - u^-$ we conclude that $u^- = u^+ - u$ is again measurable as difference of two measurable functions. (Notice that in no case '$\infty - \infty$' can occur!)

Corollary 8.12 This is trivial if the difference $u-v$ is defined. In this case it is measurable as difference of measurable functions, so

$$\{u < v\} = \{0 < u - v\}$$

etc. is measurable.

Let us be a bit more careful and consider the case where we could encounter expressions of the type '$\infty-\infty$'. Since $s_n \uparrow u$ for simple functions (they are always $\mathbb{R}$-valued...) we get

$$\{u \leqslant v\} = \{\sup_n s_n \leqslant u\} \overset{(*)}{=} \bigcap_n \{s_n \leqslant u\} = \bigcap_n \{0 \leqslant u - s_n\}$$

and the latter is a union of measurable sets, hence measurable. Now $\{u < v\} = \{u \geqslant v\}^c$ and we get measurability after switching the roles of u and v. Finally $\{u = v\} = \{u \leqslant v\} \cap \{u \geqslant v\}$ and $\{u \neq v\} = \{u = v\}^c$.

Let me stress the importance of '$\leqslant$' in $(*)$ above: we use here

$$\begin{aligned} x \in \{\sup_n s_n \leqslant u\} &\iff \sup_n s_n(x) \leqslant u(x) \\ &\overset{(**)}{\iff} s_n(x) \leqslant u(x) \quad \forall\, n \\ &\iff x \in \bigcap \{s_n \leqslant u\} \end{aligned}$$

and this would be incorrect if we had had '$<$', since the argument would break down at $(**)$ (only one implication would be valid: '$\implies$').

Problem 8.12 If u is differentiable, it is continuous, hence measurable. Moreover, since u' exists, we can write it in the form

$$u'(x) = \lim_{k\to\infty} \frac{u\big(x + \frac{1}{k}\big) - u(x)}{\frac{1}{k}}$$

i.e. as limit of measurable functions. Thus, u' is also measurable.

Problem 8.13 It is sometimes necessary to distinguish between domain and range. We use the subscript x to signal the domain, the subscript y for the range.

(i) Since $f : \mathbb{R}_x \to \mathbb{R}_y$ is $f(x) = x$, the inverse function is clearly $f^{-1}(y) = y$. So if we take any Borel set $B \in \mathcal{B}(\mathbb{R}_y)$ we get $B = f^{-1}(B) \subset \mathbb{R}_x$. Since, as we have seen, $\sigma(f) = f^{-1}(\mathcal{B}(\mathbb{R}_y))$, the above argument shows that $f^{-1}(\mathcal{B}(\mathbb{R}_y)) = \mathcal{B}(\mathbb{R}_x)$, hence $\sigma(f) = \mathcal{B}(\mathbb{R}_x)$.

(ii) The inverse map of $g(x) = x^2$ is multi-valued, i.e. if $y = x^2$, then $y = \pm\sqrt{x}$. So $g^{-1} : [0,\infty) \to \mathbb{R}$, $g^{-1}(y) = \pm\sqrt{y}$. Let us take some $B \in \mathcal{B}(\mathbb{R}_y)$. Since g^{-1} is only defined for positive numbers (squares yield positive numbers only!) we have that $g^{-1}(B) = g^{-1}(B \cap [0,\infty)) = \sqrt{B \cap [0,\infty)} \cup (-\sqrt{B \cap [0,\infty)})$ (where we used the obvious notation $\sqrt{A} = \{\sqrt{a} : a \in A\}$ and $-A = \{-a : a \in A\}$ whenever A is a set). This shows that

$$\begin{aligned}\sigma(g) &= \{\sqrt{B} \cup (-\sqrt{B}) \;:\; B \in \mathcal{B}, B \subset [0,\infty)\}\\ &= \{\sqrt{B} \cup (-\sqrt{B}) \;:\; B \in [0,\infty) \cap \mathcal{B}\}\end{aligned}$$

where we used the notation of trace σ-algebras in the latter identity.

(It is an instructive exercise to check that $\sigma(g)$ is indeed a σ-algebra. This is, of course, clear from the general theory since $\sigma(g) = g^{-1}([0,\infty) \cap \mathcal{B})$, i.e. it is the pre-image of the trace σ-algebra and pre-images of σ-algebras are always σ-algebras.

(iii) A very similar calculation as in part (ii) shows that

$$\begin{aligned}\sigma(h) &= \{B \cup (-B) \;:\; B \in \mathcal{B}, B \subset [0,\infty)\}\\ &= \{B \cup (-B) \;:\; B \in [0,\infty) \cap \mathcal{B}\}.\end{aligned}$$

(iv) As warm-up we follow the hint. The set $\{(x,y) \;:\; x + y = \alpha\}$ is the line $y = \alpha - x$ in the x-y-plane, i.e. a line with slope -1 and shift α. So $\{(x,y)\ :\ x+y \geqslant \alpha\}$ would be the points above this line and $\{(x,y)\ :\ \beta \geqslant x+y \geqslant \alpha\} = \{(x,y)\ :\ x+y \in [\alpha,\beta]\}$ would be the points in the strip which has the lines $y = \alpha - x$ and $y = \beta - x$ as boundaries.

More general, take a Borel set $B \in \mathcal{B}(\mathbb{R})$ and observe that

$$F^{-1}(B) = \{(x,y)\ :\ x+y \in B\}.$$

This set is, in an abuse of notation, $y = B - x$, i.e. these are all lines with slope -1 (135 degrees) and *every possible shift from the set B*—it gives a kind of stripe-pattern. To sum up:

$\sigma(F) = \{$all 135-degree diagonal stripes in $\mathbb{R}^2$ with 'base' $B \in \mathcal{B}(\mathbb{R})\}$.

(v) Again follow the hint to see that $\{(x,y)\ :\ x^2+y^2 = r\}$ is a circle, radius r, centre $(0,0)$. So $\{(x,y)\ :\ x^2+y^2 \leqslant r\}$ is the solid disk, radius r, centre $(0,0)$ and $\{(x,y)\ :\ R \geqslant x^2+y^2 \geqslant r\} = \{(x,y)\ :\ x^2+y^2 \in [r,R]\}$ is the annulus with exterior radius R and interior radius r about $(0,0)$.

More general, take a Borel set $B \subset [0,\infty), B \in \mathcal{B}(\mathbb{R})$, i.e. $B \in [0,\infty) \cap \mathcal{B}(\mathbb{R})$ (negative radii don't make sense!) and observe that the set $\{(x,y)\ :\ x^2+y^2 \in B\}$ gives a ring-pattern which is 'supported' by the set B (i.e. we take all circles passing through B...). To sum up:

$$\begin{aligned}\sigma(G) = \{&\text{a set consists of all circles in } \mathbb{R}^2 \text{ about } (0,0)\\ &\text{passing through } B \in [0,\infty) \cap B(\mathbb{R})\}.\end{aligned}$$

Problem 8.14 Assume first that u is injective. This means that every point in the range $u(\mathbb{R})$ comes exactly from one uniquely defined $x \in \mathbb{R}$. This can be expressed by saying that $\{x\} = u^{-1}(\{u(x)\})$ — but the singleton $\{u(x)\}$ is a Borel set in the range, so $\{x\} \in \sigma(u)$ as $\sigma(u) = u^{-1}(u(\mathbb{R}) \cap \mathcal{B})$.

Conversely, assume that for each x we have $\{x\} \in \sigma(u)$. Fix an x_0 and call $u(x_0) = \alpha$. Since u is measurable, the set $\{u = \alpha\} = \{x : u(x) = \alpha\}$ is measurable and, clearly, $\{x_0\} \subset \{u = \alpha\}$. But if we had another $x_0 \neq x_1 \in \{u = \alpha\}$ this would mean that we could never 'produce' $\{x_0\}$ on its own as a pre-image of some set, but we must be able to do so as $\{x_0\} \in \sigma(u)$, by assumption. Thus, $x_1 = x_0$. To sum up, we have shown that $\{u = \alpha\}$ consists of one point only, i.e. we have shown that $u(x_0) = u(x_1)$ implies $x_0 = x_1$ which is just injectivity.

Problem 8.15 Clearly $u : \mathbb{R} \to [0,\infty)$. So let's take $I = (a,b) \subset [0,\infty)$. Then $u^{-1}((a,b)) = (-b,-a) \cup (a,b)$. This shows that for $\mu := \lambda \circ u^{-1}$

$$\begin{aligned}\mu(a,b) &= \lambda \circ u^{-1}((a,b)) = \lambda\big((-b,-a) \cup (a,b)\big) = \lambda(-b,-a) + \lambda(a,b)\\ &= (-a-(-b)) + (b-a) = 2(b-a) = 2\lambda((a,b)).\end{aligned}$$

This shows that $\mu = 2\lambda$ if we allow only intervals from $[0,\infty)$, i.e.

$$\mu(I) = 2\lambda\big(I \cap [0,\infty)\big) \quad \text{for any interval } I \subset \mathbb{R}.$$

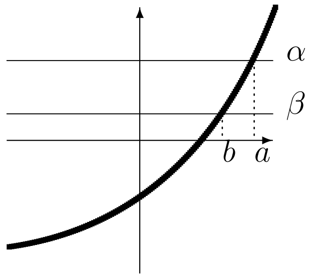
Since a measure on the Borel sets is completely described by (either: open or closed or half-open or half-closed) intervals (the intervals generate the Borel sets!), we can invoke the uniqueness theorem to guarantee that the above equality holds for all Borel sets.

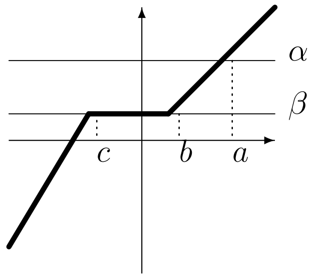
Problem 8.16

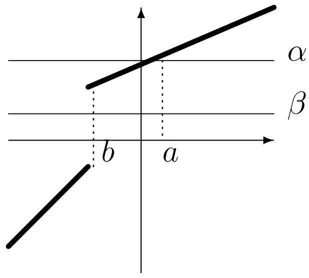
- clear, since $u(x-2)$ is a combination of the measurable shift τ_2 and the measurable function u.
- this is trivial since $u \mapsto e^u$ is a continuous function, as such it is measurable and combinations of measurable functions are again measurable.
- this is trivial since $u \mapsto \sin(u+8)$ is a continuous function, as such it is measurable and combinations of measurable functions are again measurable.
- iterate Problem 8.12
- obviously, $\operatorname{sgn} x = (-1) \cdot \mathbf{1}_{(-\infty,0)}(x) + 0 \cdot \mathbf{1}_{\{0\}}(x) + 1 \cdot \mathbf{1}_{(0,\infty)}(x)$, i.e. a measurable function. Using the first example, we see now that $\operatorname{sgn} u(x-7)$ is a combination of three measurable functions.

Problem 8.17 Let $A \subset \mathbb{R}$ be such that $A \notin \mathcal{B}$. Then it is clear that $u(x) = \mathbf{1}_A(x) - \mathbf{1}_{A^c}(x)$ is NOT measurable (take, e.g. $A = \{f = 1\}$ which should be measurable for measurable functions), but clearly, $|f(x)| = 1$ and as constant function this IS measurable.

Problem 8.18 We want tho show that the sets $\{u \leqslant \alpha\}$ are Borel sets. We will even show that they are intervals, hence Borel sets. Imagine the graph of an increasing function and the line $y = \alpha$ cutting through. Essentially we have three scenarios: the cut happens at a point where (a) u is continuous and strictly increasing or (b) u is flat or (c) u jumps—i.e. has a gap; these three cases are shown in the following pictures:







From the three pictures it is clear that we get in any case an interval for the sub-level sets $\{u \leqslant \gamma\}$ where γ is some level (in the pic's $\gamma = \alpha$ or $= \beta$), you can read off the intervals on the abscissa where the dotted lines cross the abscissa.

Now let's look at the additional conditions: First the intuition: From the first picture, the continuous and strictly increasing case, it is clear that we can produce any interval $(-\infty, b]$ to $(-\infty, a]$ by looking at $\{u \leqslant \beta\}$ to $\{u \leqslant \alpha\}$ my moving up the β-line to level α. The point is here that we get all intervals, so we get a generator of the Borel sets, so we should get all Borel sets.

The second picture is bad: the level set $\{u \leqslant \beta\}$ is $(-\infty, b]$ and all level sets below will only come up to the point $(-\infty, c]$, so there is no chance to get any set contained in (c, b), i.e. we cannot get all Borel sets.

The third picture is good again, because the vertical jump does not hurt. The only 'problem' is whether $\{u \leqslant \beta\}$ is $(-\infty, b]$ or $(-\infty, b)$ which essentially depends on the property of the graph whether $u(b) = \beta$ or not, but this is not so relevant here, we just must make sure that we can get more or less all intervals. The reason, really, is that jumps as we described them here can only happen countably often, so this problem occurs only countably often, and we can overcome it therefore.

So the point is: we must disallow flat bits, i.e. $\sigma(u)$ is the Borel σ-algebra if, and only, if u is strictly increasing, i.e. if, and only if, u is injective. (Note that this would have been clear already from Problem 8.14, but our approach here is much more intuitive.)