9 Integration of positive functions. Solutions to Problems 9.1–9.12

Problem 9.1 We know that for any two simple functions $f, g \in \mathcal{E}_+$ we have $I_\mu(f+g) = I_\mu(f) + I_\mu(g)$ (=additivity), and this is easily extended to finitely many, say, m different positive simple functions. Observe now that each $\xi_j \mathbf{1}_{A_j}$ is a positive simple function, hence

$$I_\mu\left(\sum_{j=1}^m \xi_j \mathbf{1}_{A_j}\right) = \sum_{j=1}^m I_\mu\left(\xi_j \mathbf{1}_{A_j}\right) = \sum_{j=1}^m \xi_j I_\mu\left(\mathbf{1}_{A_j}\right) = \sum_{j=1}^m \xi_j \mu\left(A_j\right).$$

Put in other words: we have used the linearity of I_μ.

Problem 9.2 We check Properties 9.8(i)–(iv).

(i) This follows from Properties 9.3 and Lemme 9.5 since $\int \mathbf{1}_A \, d\mu = I_\mu(\mathbf{1}_A) = \mu(A)$.

(ii) This follows again from Properties 9.3 and Corollary 9.7 since for $u_n \in \mathcal{E}_+$ with $u = \sup_n u_n$ (note: the sup's are increasing limits!) we have

$$\begin{aligned}
\int \alpha u \, d\mu = \int \alpha \sup_n u_n \, d\mu &= \sup_n I_\mu(\alpha u_n) \\
&= \sup_n \alpha I_\mu(u_n) \\
&= \alpha \sup_n I_\mu(u_n) \\
&= \alpha \int u \, d\mu.
\end{aligned}$$

(iii) This follows again from Properties 9.3 and Corollary 9.7 since for $u_n, v_n \in \mathcal{E}_+$ with $u = \sup_n u_n, v = \sup_n v_n$ (note: the sup's are increasing limits!) we have

$$\begin{aligned}
\int (u+v) \, d\mu = \int \lim_{n\to\infty} (u_n + v_n) \, d\mu &= \lim_{n\to\infty} I_\mu(u_n + v_n) \\
&= \lim_{n\to\infty} \left(I_\mu(u_n) + I_\mu(v_n)\right) \\
&= \lim_{n\to\infty} I_\mu(u_n) + \lim_{n\to\infty} I_\mu(v_n) \\
&= \int u \, d\mu + \int v \, d\mu.
\end{aligned}$$

(iv) This was shown in step 1 of the proof of the Beppo Levi theorem 9.6

Problem 9.3 Consider on the space $([-1,0],\lambda)$, $\lambda(dx) = dx$ is Lebesgue measure on $[0,1]$, the sequence of 'tent-type' functions

$$f_k(x) = \begin{cases} 0, & -1 \leqslant x \leqslant -\frac{1}{k}, \\ k^3\big(x + \frac{1}{k}\big), & -\frac{1}{k} \leqslant x \leqslant 0, \end{cases} \qquad (k \in \mathbb{N}),$$

(draw a picture!). These are clearly monotonically increasing functions but, as a sequence, we do not have $f_k(x) \leqslant f_{k+1}(x)$ for *every* x! Note also that each function is integrable (with integral $\frac{1}{2}k$) but the pointwise limit is not integrable.

Problem 9.4 Following the hint we set $s_m = u_1 + u_2 + \ldots + u_m$. As a finite sum of positive measurable functions this is again positive and measurable. Moreover, s_m increases to $s = \sum_{j=1}^\infty u_j$ as $m \to \infty$. Using the additivity of the integral (9.8 (iii)) and the Beppo Levi theorem 9.6 we get

$$\begin{aligned} \int \sum_{j=1}^\infty u_j \, d\mu = \int \sup_m s_m \, d\mu &= \sup_m \int s_m \, d\mu \\ &= \sup_m \int (u_1 + \ldots + u_m) \, d\mu \\ &= \sup_m \sum_{j=1}^m \int u_j \, d\mu \\ &= \sum_{j=1}^\infty \int u_j \, d\mu. \end{aligned}$$

Conversely, assume that 9.9 is true. We want to deduce from it the validity of Beppo Levi's theorem 9.6. So let $(w_j)_{j\in\mathbb{N}}$ be an increasing sequence of measurable functions with limit $w = \sup_j w$. For ease of notation we set $w_0 \equiv 0$. Then we can write each w_j as a partial sum

$$w_j = (w_j - w_{j-1}) + \cdots + (w_1 - w_0)$$

of positive measurable summands of the form $u_k := w_k - w_{k-1}$. Thus,

$$w_m = \sum_{k=1}^m u_k \quad \text{and} \quad w = \sum_{k=1}^\infty u_k$$

and, using the additivity of the integral,

$$\int w\,d\mu \overset{9.9}{=} \sum_{k=1}^{\infty} \int u_k\,d\mu = \sup_m \int \sum_{k=1}^{m} u_k\,d\mu = \sup_m \int w_m\,d\mu.$$

Problem 9.5 Set $\nu(A) := \int \mathbf{1}_A u\,d\mu$. Then ν is a $[0,\infty]$-valued set-function defined for $A \in \mathcal{A}$.

(M_1) Since $\mathbf{1}_\emptyset \equiv 0$ we have clearly $\nu(\emptyset) = \int 0 \cdot u\,d\mu = 0$.

(M_1) Let $A = \dot{\bigcup}_{j\in\mathbb{N}} A_j$ a disjoint union of sets $A_j \in \mathcal{A}$. Then

$$\sum_{j=1}^{\infty} \mathbf{1}_{A_j} = \mathbf{1}_A$$

and we get from Corollary 9.9

$$\begin{aligned}\nu(A) = \int \left(\sum_{j=1}^{\infty} \mathbf{1}_{A_j}\right) \cdot u\,d\mu &= \int \sum_{j=1}^{\infty} \big(\mathbf{1}_{A_j} \cdot u\big)\,d\mu \\ &= \sum_{j=1}^{\infty} \int \mathbf{1}_{A_j} \cdot u\,d\mu \\ &= \sum_{j=1}^{\infty} \nu(A_j).\end{aligned}$$

Problem 9.6 This is actually trivial: since our σ-algebra is $\mathcal{P}(\mathbb{N})$, all subsets of $\mathbb{N}$ are measurable. Now the sub-level sets $\{u \leqslant \alpha\} = \{k \in \mathbb{N} : u(k) \leqslant \alpha\}$ are always $\subset \mathbb{N}$ and as such they are $\in \mathcal{P}(\mathbb{N})$, hence u is always measurable.

Problem 9.7 We have seen in Problem 4.6 that μ is indeed a measure. We follow the instructions. First, for $A \in \mathcal{A}$ we get

$$\int \mathbf{1}_A\,d\mu = \mu(A) = \sum_{j\in\mathbb{N}} \mu_j(A) = \sum_{j\in\mathbb{N}} \int \mathbf{1}_A\,d\mu_j.$$

By the linearity of the integral, this easily extends to functions of the form $\alpha\mathbf{1}_A + \beta\mathbf{1}_B$ where $A, B \in \mathcal{A}$ and $\alpha, \beta \geqslant 0$:

$$\int (\alpha\mathbf{1}_A + \beta\mathbf{1}_B)\,d\mu = \alpha \int \mathbf{1}_A\,d\mu + \beta \int \mathbf{1}_B\,d\mu$$

$$= \alpha \sum_{j\in\mathbb{N}} \int \mathbf{1}_A \, d\mu_j + \beta \sum_{j\in\mathbb{N}} \int \mathbf{1}_B \, d\mu_j$$
$$= \sum_{j\in\mathbb{N}} \int (\alpha \mathbf{1}_A + \beta \mathbf{1}_B) \, d\mu_j$$

and this extends obviously to simple functions which are finite sums of the above type.

$$\int f \, d\mu = \sum_{j\in\mathbb{N}} \int f \, d\mu_j \qquad \forall f \in \mathcal{E}_+.$$

Finally, take $u \in \mathcal{M}_+$ and take an approximating sequence $u_n \in \mathcal{E}_+$ with $\sup_n u_n = u$. Then we get by Beppo Levi (indicated by an asterisk $*$)

$$\begin{aligned}
\int u \, d\mu &\overset{*}{=} \sup_n \int u_n \, d\mu = \sup_n \sum_{j=1}^{\infty} \int u_n \, d\mu_j \\
&= \sup_n \sup_m \sum_{j=1}^{m} \int u_n \, d\mu_j \\
&= \sup_m \sup_n \sum_{j=1}^{m} \int u_n \, d\mu_j \\
&= \sup_m \lim_n \sum_{j=1}^{m} \int u_n \, d\mu_j \\
&= \sup_m \sum_{j=1}^{m} \lim_n \int u_n \, d\mu_j \\
&\overset{*}{=} \sup_m \sum_{j=1}^{m} \int \lim_n u_n \, d\mu_j \\
&= \sum_{j=1}^{\infty} \int u \, d\mu_j
\end{aligned}$$

where we repeatedly used that all sup's are increasing limits and that we may swap any two sup's (this was the hint to the hint to Problem 4.6.)

Problem 9.8 Set $w_j := u - u_j$. Then the w_j are a sequence of positive measurable functions. By Fatou's lemma we get

$$\int \liminf_j w_j \, d\mu \leqslant \liminf_j \int w_j \, d\mu$$

$$= \liminf_j \left(\int u \, d\mu - \int u_j \, d\mu \right)$$
$$= \int u \, d\mu - \limsup_j \int u_j \, d\mu$$

(see, e.g. the rules for lim inf and lim sup in Appendix A). Thus,

$$\begin{aligned}\int u \, d\mu - \limsup_j \int u_j \, d\mu &\geqslant \int \liminf_j w_j \, d\mu \\ &= \int \liminf_j (u - u_j) \, d\mu \\ &= \int \big(u - \limsup_j u_j\big) \, d\mu\end{aligned}$$

and the claim follows by subtracting the *finite* value $\int u \, d\mu$ on both sides.

Remark. The uniform domination of u_j by an integrable function u is really important. Have a look at the following situation: $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, $\lambda(dx) = dx$ denotes Lebesgue measure, and consider the positive measurable functions $u_j(x) = \mathbf{1}_{[j,2j]}(x)$. Then $\limsup_j u_j(x) = 0$ but $\limsup_j \int u_j \, d\lambda = \limsup_j j = \infty \neq \int 0 \, d\lambda$.

Problem 9.9 (i) Have a look at Appendix A, Lemma A.2.

(ii) You have two possibilities: the set-theoretic version:

$$\begin{aligned}\mu\big(\liminf_j A_j\big) &= \mu\Big(\bigcup_k \bigcap_{j\geqslant k} A_j\Big) \\ &\overset{*}{=} \sup_k \underbrace{\mu\Big(\bigcap_{j\geqslant k} A_j\Big)}_{\substack{\leqslant \mu(A_j)\ \forall j\geqslant k \\ \text{hence, } \leqslant \inf_{j\geqslant k} \mu(A_j)}} \\ &\leqslant \sup_k \inf_{j\geqslant k} \mu(A_j) \\ &= \liminf_j \mu(A_j)\end{aligned}$$

which uses at the point $*$ the continuity of measures, Theorem 4.4.

The *alternative* would be (i) combined with Fatou's lemma:

$$\mu\big(\liminf_j A_j\big) = \int \mathbf{1}_{\liminf_j A_j} \, d\mu$$

$$
\begin{aligned}
&= \int \liminf_j \mathbf{1}_{A_j}\,d\mu \\
&\leqslant \liminf_j \int \mathbf{1}_{A_j}\,d\mu
\end{aligned}
$$

(iii) Again, you have two possibilities: the set-theoretic version:

$$
\begin{aligned}
\mu\big(\limsup_j A_j\big) &= \mu\Big(\bigcap_k \bigcup_{j\geqslant k} A_j\Big) \\
&\overset{\#}{=} \inf_k \quad \underbrace{\mu\Big(\bigcup_{j\geqslant k} A_j\Big)}_{\substack{\geqslant \mu(A_j)\ \forall j\geqslant k \\ \text{hence, } \geqslant \sup_{j\geqslant k}\mu(A_j)}} \\
&\geqslant \inf_k \sup_{j\geqslant k} \mu(A_j) \\
&= \limsup_j \mu(A_j)
\end{aligned}
$$

which uses at the point # the continuity of measures, Theorem 4.4. This step uses the finiteness of μ.

The *alternative* would be (i) combined with the reversed Fatou lemma of Problem 9.8:

$$
\begin{aligned}
\mu\big(\limsup_j A_j\big) &= \int \mathbf{1}_{\limsup_j A_j}\,d\mu \\
&= \int \limsup_j \mathbf{1}_{A_j}\,d\mu \\
&\geqslant \limsup_j \int \mathbf{1}_{A_j}\,d\mu
\end{aligned}
$$

(iv) Take the example in the remark to the solution for Problem 9.8. We will discuss it here in its set-theoretic form: take $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ with λ denoting Lebesgue measure $\lambda(dx) = dx$. Put $A_j = [j, 2j] \in \mathcal{B}(\mathbb{R})$. Then

$$
\limsup_j A_j = \bigcap_k \bigcup_{j\geqslant k} [j, 2j] = \bigcap_k [k, \infty) = \emptyset
$$

But $0 = \lambda(\emptyset) \geqslant \limsup_j \lambda(A_j) = \limsup_j j = \infty$ is a contradiction. (The problem is that $\lambda[k, \infty) = \infty$!)

Problem 9.10 We use the fact that, because of disjointness,

$$1 = \mathbf{1}_X = \sum_{j=1}^{\infty} \mathbf{1}_{A_j}$$

so that, because of Corollary 9.9,

$$\begin{aligned}\int u\,d\mu = \int \Big(\sum_{j=1}^{\infty} \mathbf{1}_{A_j}\Big) \cdot u\,d\mu &= \int \sum_{j=1}^{\infty} \big(\mathbf{1}_{A_j} \cdot u\big)\,d\mu \\ &= \sum_{j=1}^{\infty} \int \mathbf{1}_{A_j} \cdot u\,d\mu.\end{aligned}$$

Assume now that $(X, \mathcal{A}, \mu)$ is σ-finite with an exhausting sequence of sets $(B_j)_j \subset \mathcal{A}$ such that $B_j \uparrow X$ and $\mu(B_j) < \infty$. Then we make the B_j's pairwise disjoint by setting

$$A_1 := B_1, \qquad A_k := B_k \setminus (B_1 \cup \cdots \cup B_{k-1}) = B_k \setminus B_{k-1}.$$

Now take any sequence $(a_k)_k \subset (0, \infty)$ with $\sum_k a_k \mu(A_k) < \infty$—e.g. $a_k := 2^{-k}/(\mu(A_k) + 1)$—and put

$$w(x) := \sum_{j=1}^{\infty} a_k \mathbf{1}_{A_k}.$$

Then w is integrable and, obviously, $w(x) > 0$ everywhere.

Problem 9.11 (i) We check $(M_1), (M_2)$. Using the fact that $N(x, \cdot)$ is a measure, we find

$$\mu N(\emptyset) = \int N(x, \emptyset)\,\mu(dx) = \int 0\,\mu(dx) = 0.$$

Further, let $(A_j)_{j \in \mathbb{N}} \subset \mathcal{A}$ be a sequence of disjoint sets and set $A = \dot\bigcup_j A_j$. Then

$$\begin{aligned}\mu N(A) = \int N\Big(x, \dot\bigcup_j A_j\Big)\,\mu(dx) &= \int \sum_j N(x, A_j)\,\mu(dx) \\ &\overset{9.9}{=} \sum_j \int N(x, A_j)\,\mu(dx) \\ &= \sum_j \mu N(A_j).\end{aligned}$$

(ii) We have for $A, B \in \mathcal{A}$ and $\alpha, \beta \geqslant 0$,

$$\begin{aligned} N(\alpha \mathbf{1}_A + \beta \mathbf{1}_B)(x) &= \int \big(\alpha \mathbf{1}_A(y) + \beta \mathbf{1}_B(y)\big)\, N(x, dy) \\ &= \alpha \int \mathbf{1}_A(y)\, N(x, dy) + \beta \int \mathbf{1}_B(y)\, N(x, dy) \\ &= \alpha N \mathbf{1}_A(x) + \beta N \mathbf{1}_B(x). \end{aligned}$$

Thus $N(f+g)(x) = Nf(x) + Ng(x)$ for positive simple $f, g \in \mathcal{E}^+(\mathcal{A})$. Moreover, since by Beppo Levi (marked by an asterisk $*$) for an increasing sequence $f_k \uparrow u$

$$\begin{aligned} \sup_k Nf_k(x) = \sup_k \int f_k(y)\, N(x, dy) &\overset{*}{=} \int \sup_k f_k(y)\, N(x, dy) \\ &= \int u(y)\, N(x, dy) \\ &= Nu(x) \end{aligned}$$

and since the sup is actually an increasing limit, we see for positive measurable $u, v \in \mathcal{M}^+(\mathcal{A})$ and the corresponding increasing approximations via positive simple functions f_k, g_k:

$$\begin{aligned} N(u+v)(x) &= \sup_k N(f_k + g_k)(x) \\ &= \sup_k Nf_k(x) + \sup_k Ng_k(x) \\ &= Nu(x) + Nv(x). \end{aligned}$$

Moreover, $x \mapsto N\mathbf{1}_A(x) = N(x, A)$ is a measurable function, thus $Nf(x)$ is a measurable function for all simple $f \in \mathcal{E}^+(\mathcal{A})$ and, by Beppo Levi (see above) $Nu(x)$, $u \in \mathcal{M}^+(\mathcal{A})$, is for every x an increasing limit of measurable functions $Nf_k(x)$. Therefore, $Nu \in \mathcal{M}^+(\mathcal{A})$.

(iii) If $u = \mathbf{1}_A$, $A \in \mathcal{A}$, we have

$$\begin{aligned} \int \mathbf{1}_A(y)\, \mu N(dy) = \mu N(A) &= \int N(x, A)\, \mu(dx) \\ &= \int N\mathbf{1}_A(x)\, \mu(dx). \end{aligned}$$

By linearity this carries over to $f \in \mathcal{E}^+(\mathcal{A})$ and, by a Beppo-Levi argument, to $u \in \mathcal{M}^+(\mathcal{A})$.

Problem 9.12 Put

$$\nu(A) := \int u \cdot \mathbf{1}_{A_\sigma^+}\,d\mu + \int (1-u)\cdot \mathbf{1}_{A_\sigma^-}\,d\mu.$$

If A is symmetric w.r.t. the origin, $A^+ = -A^-$ and $A_\sigma^\pm = A$. Therefore,

$$\nu(A) = \int u \cdot \mathbf{1}_A\,d\mu + \int (1-u)\cdot \mathbf{1}_A\,d\mu = \int \mathbf{1}_A\,d\mu = \mu(A).$$

This means that ν extends μ. It also shows that $\nu(\emptyset) = 0$. Since ν is defined for all sets from $\mathcal{B}(\mathbb{R})$ and since ν has values in $[0,\infty]$, it is enough to check σ-additivity.

For this, let $(A_j)_j \subset \mathcal{B}(\mathbb{R})$ be a sequence of pairwise disjoint sets. From the definitions it is clear that the sets $(A_j)_\sigma^\pm$ are again pairwise disjoint and that $\biguplus_j (A_j)_\sigma^\pm = \big(\biguplus_j A_j\big)_\sigma^\pm$. Since each of the set-functions

$$B \mapsto \int u \cdot \mathbf{1}_B\,d\mu, \qquad C \mapsto \int (1-u)\cdot \mathbf{1}_C\,d\mu$$

is σ-additive, it is clear that their sum ν will be σ-additive, too.

The obvious non-uniqueness of the extension does not contradict the uniqueness theorem for extensions, since Σ does not generate $\mathcal{B}(\mathbb{R})$!

10 Integrals of measurable functions and null sets. Solutions to Problems 10.1–10.16

Problem 10.1 Let u, v be integrable functions and $a, b \in \mathbb{R}$. Assume that either u, v are real-valued or that $au + bv$ makes sense (i.e. avoiding the case '$\infty - \infty$'). Then we have

$$|au + bv| \leqslant |au| + |bv| = |a| \cdot |u| + |b| \cdot |v| \leqslant K(|u| + |v|)$$

with $K = \max\{|a|, |b|\}$. Since the RHS is integrable (because of Theorem 10.3 and Properties 9.8) we have that $au + bv$ is integrable by Theorem 10.3. So we get from Theorem 10.4 that

$$\int (au + bv)\, d\mu = \int au\, d\mu + \int bv\, d\mu = a \int u\, d\mu + b \int v\, d\mu$$

and this is what was claimed.

Problem 10.2 We follow the hint and show first that $u(x) := x^{-1/2}$, $0 < x < 1$, is Lebesgue integrable. The idea here is to construct a sequence of simple functions approximating u from below. Define

$$u_n(x) := \begin{cases} 0, & \text{if } x \in (0, \frac{1}{n}) \\ u(\frac{j+1}{n}), & \text{if } x \in [\frac{j}{n}, \frac{j+1}{n}), \quad j = 1, \dots n-1 \end{cases}$$

$$\iff \qquad u_n = \sum_{j=1}^{n-1} u(\tfrac{j+1}{n}) \mathbf{1}_{\left[\frac{j}{n}, \frac{j+1}{n}\right)}$$

which is clearly a simple function. Also $u_n \leqslant u$ and $\lim_{n\to\infty} u_n(x) = \sup_n u_n(x) = u(x)$ for all x.

Since $P(A)$ is just $\lambda(A \cap (0, 1))$, the integral of u_n is given by

$$\begin{aligned} \int u_n\, dP = I_P(u_n) &= \sum_{j=1}^{n-1} u(\tfrac{j+1}{n}) \lambda[\tfrac{j}{n}, \tfrac{j+1}{n}) \\ &= \sum_{j=1}^{n-1} \sqrt{\tfrac{j+1}{n}} \cdot \tfrac{1}{n} \end{aligned}$$

$$\leqslant \sum_{j=1}^{n-1} \tfrac{1}{n} \leqslant 1$$

and is thus finite, even uniformly in n! So, Beppo Levi's theorem tells us that

$$\int u\,dP = \sup_n \int u_n\,dP \leqslant \sup_n 1 = 1 < \infty$$

showing integrability.

Now u is clearly not bounded but integrable.

Problem 10.3 True, we can change an integrable function on a null set, even by setting it to the value $+\infty$ or $-\infty$ *on the null set*. This is just the assertion of Theorem 10.9 and its Corollaries 10.10, 10.11.

Problem 10.4 We have seen that a single point is a Lebesgue null set: $\{x\} \in \mathcal{B}(\mathbb{R})$ for all $x \in \mathbb{R}$ and $\lambda(\{x\}) = 0$, see e.g. Problems 4.11 and 6.4. If N is countable, we know that $N = \{x_j : j \in \mathbb{N}\} = \dot{\bigcup}_{j\in\mathbb{N}}\{x_j\}$ and by the σ-additivity of measures

$$\lambda(N) = \lambda\left(\dot{\bigcup_{j\in\mathbb{N}}}\{x_j\}\right) = \sum_{j\in\mathbb{N}} \lambda(\{x_j\}) = \sum_{j\in\mathbb{N}} 0 = 0.$$

The Cantor set C from Problem 7.10 is, as we have seen, uncountable but has measure $\lambda(C) = 0$. This means that there are uncountable sets with measure zero.

In $\mathbb{R}^2$ and for two-dimensional Lebesgue measure λ^2 the situation is even easier: every line L in the plane has zero Lebesgue measure and L contains certainly uncountably many points. That $\lambda^2(L) = 0$ is seen from the fact that L differs from the ordinate $\{(x, y) \in \mathbb{R}^2 : x = 0\}$ only by a rigid motion T which leaves Lebesgue measure invariant (see Chapter 5) and $\lambda^2(\{x = 0\}) = 0$ as seen in Problem 6.4.

Problem 10.5 (i) Since $\{|u| > c\} \subset \{|u| \geqslant c\}$ and, therefore, $\mu(\{|u| > c\}) \leqslant \mu(\{|u| \geqslant c\})$, this follows immediately from Proposition 10.12. Alternatively, one could also mimic the proof of this Proposition or use part (iii) of the present problem with $\phi(t) = t$, $t \geqslant 0$.

(ii) This will follow from (iii) with $\phi(t) = t^p$, $t \geqslant 0$, since $\mu(\{|u| > c\}) \leqslant \mu(\{|u| \geqslant c\})$ as $\{|u| > c\} \subset \{|u| \geqslant c\}$.

(iii) We have, since ϕ is increasing,

$$\mu(\{|u| \geqslant c\}) = \mu(\{\phi(|u|) \geqslant \phi(c)\})$$

$$
\begin{aligned}
&= \int \mathbf{1}_{\{x\,:\,\phi(|u(x)|)\geqslant\phi(c)\}}(x)\,\mu(dx)\\
&= \int \frac{\phi(|u(x)|)}{\phi(|u(x)|)}\,\mathbf{1}_{\{x\,:\,\phi(|u(x)|)\geqslant\phi(c)\}}(x)\,\mu(dx)\\
&\leqslant \int \frac{\phi(|u(x)|)}{\phi(c)}\,\mathbf{1}_{\{x\,:\,\phi(|u(x)|)\geqslant\phi(c)\}}(x)\,\mu(dx)\\
&\leqslant \int \frac{\phi(|u(x)|)}{\phi(c)}\,\mu(dx)\\
&= \frac{1}{\phi(c)}\int \phi(|u(x)|)\,\mu(dx)
\end{aligned}
$$

(iv) Let us set $b = \alpha\int u\,d\mu$. Then we follow the argument of (iii):

$$
\begin{aligned}
\mu(\{u \geqslant b\}) &= \int \mathbf{1}_{\{x\,:\,u(x)\geqslant b\}}(x)\,\mu(dx)\\
&= \int \frac{u(x)}{u(x)}\,\mathbf{1}_{\{x\,:\,u(x)\geqslant b\}}(x)\,\mu(dx)\\
&\leqslant \int \frac{u(x)}{b}\,\mathbf{1}_{\{x\,:\,u(x)\geqslant b\}}(x)\,\mu(dx)\\
&\leqslant \int \frac{u}{b}\,d\mu\\
&= \frac{1}{b}\int u\,d\mu
\end{aligned}
$$

and substituting $\alpha\int u\,d\mu$ for b shows the inequality.

(v) Using the fact that ψ is decreasing we get $\{|u| < c\} = \{\psi(|u|) > \psi(c)\}$—mind the change of the inequality sign—and going through the proof of part (iii) again we used there that ϕ increases only in the first step in a similar role as we used the decrease of ψ here! This means that the argument of (iii) is valid after this step and we get, altogether,

$$
\begin{aligned}
\mu(\{|u| < c\}) &= \mu(\{\psi(|u|) > \psi(c)\})\\
&= \int \mathbf{1}_{\{x\,:\,\psi(|u(x)|)>\psi(c)\}}(x)\,\mu(dx)\\
&= \int \frac{\psi(|u(x)|)}{\psi(|u(x)|)}\,\mathbf{1}_{\{x\,:\,\psi(|u(x)|)>\phi(c)\}}(x)\,\mu(dx)\\
&\leqslant \int \frac{\psi(|u(x)|)}{\psi(c)}\,\mathbf{1}_{\{x\,:\,\psi(|u(x)|)>\psi(c)\}}(x)\,\mu(dx)
\end{aligned}
$$

$$\leqslant \int \frac{\psi(|u(x)|)}{\psi(c)}\,\mu(dx)$$
$$= \frac{1}{\psi(c)} \int \psi(|u(x)|)\,\mu(dx)$$

(vi) This follows immediately from (ii) by taking $\mu = P$, $c = \alpha\sqrt{VX}$, $u = X - EX$ and $p = 2$. Then

$$P(|X - EX| \geqslant \alpha EX) \leqslant \frac{1}{(\alpha\sqrt{VX})^2} \int |X - EX|^2\,dP$$
$$= \frac{1}{\alpha^2 VX}\,VX \;=\; \frac{1}{\alpha^2}.$$

Problem 10.6 We mimic the proof of Corollary 10.13. Set $N = \{|u| = \infty\} = \{|u|^p = \infty\}$. Then $N = \bigcap_{k\in\mathbb{N}}\{|u|^p \geqslant k\}$ and using Markov's inequality (MI) and the 'continuity' of measures, Theorem 4.4, we find

$$\mu(N) = \mu\left(\bigcap_{k\in\mathbb{N}}\{|u|^p \geqslant k\}\right) \overset{4.4}{=} \lim_{k\to\infty} \mu(\{|u|^p \geqslant k\})$$
$$\overset{MI}{\leqslant} \lim_{k\to\infty} \frac{1}{k} \underbrace{\int |u|^p\,d\mu}_{<\infty} \;=\; 0.$$

For arctan this is not any longer true for several reasons:

- ... arctan is odd and changes sign, so there could be cancelations under the integral.
- ... even if we had no cancelations we have the problem that the points where $u(x) = \infty$ are now transformed to points where $\arctan(u(x)) = \frac{\pi}{2}$ and we do not know how the measure μ acts under this transformation. A simple example: Take μ to be a measure of total finite mass (that is: $\mu(X) < \infty$), e.g. a probability measure, and take the function $u(x)$ which is constantly $u \equiv +\infty$. Then $\arctan(u(x)) = \frac{\pi}{2}$ throughout, and we get

$$\int \arctan u(x)\,\mu(dx) = \int \frac{\pi}{2}\,d\mu = \frac{\pi}{2}\int d\mu = \frac{\pi}{2}\,\mu(X) < \infty,$$

but u is *nowhere* finite!

Problem 10.7 '$\Longrightarrow$': since the A_j are disjoint we get the identities

$$\mathbf{1}_{\bigcup_j A_j} = \sum_{k=1}^{\infty} \mathbf{1}_{A_j} \qquad \text{and so} \qquad u \cdot \mathbf{1}_{\bigcup_j A_j} = \sum_{k=1}^{\infty} u \cdot \mathbf{1}_{A_j},$$

hence $|u\mathbf{1}_{A_n}| = |u|\mathbf{1}_{A_n} \leqslant |u|\mathbf{1}_{\bigcup_j A_j} = |u\mathbf{1}_{\bigcup_j A_j}|$ showing the integrability of each $u\mathbf{1}_{A_n}$ by Theorem 10.3. By a Beppo Levi argument (Theorem 9.6) or, directly, by Corollary 9.9 we get

$$\begin{aligned}\sum_{j=1}^{\infty} \int_{A_j} |u|\,d\mu = \sum_{j=1}^{\infty} \int |u|\mathbf{1}_{A_j}\,d\mu &= \int \sum_{j=1}^{\infty} |u|\mathbf{1}_{A_j}\,d\mu \\ &= \int |u|\mathbf{1}_{\bigcup_j A_j}\,d\mu \; < \; \infty.\end{aligned}$$

The converse direction '$\Longleftarrow$' follows again from Corollary 9.9, now just the other way round:

$$\begin{aligned}\int |u|\mathbf{1}_{\bigcup_j A_j}\,d\mu = \int \sum_{j=1}^{\infty} |u|\mathbf{1}_{A_j}\,d\mu &= \sum_{j=1}^{\infty} \int |u|\mathbf{1}_{A_j}\,d\mu \\ &= \sum_{j=1}^{\infty} \int_{A_j} |u|\,d\mu \; < \; \infty\end{aligned}$$

showing that $u\mathbf{1}_{\bigcup_j A_j}$ is integrable.

Problem 10.8 One possibility to solve the problem is to follow the hint. We go here a different (shorter) direction.

(i) Observe that $u_j - v \geqslant 0$ is a sequence of positive and integrable functions. Applying Fatou's lemma (in the usual form) yields (observing the rules for lim inf, lim sup from Appendix A, compare also Problem 9.8):

$$\begin{aligned}\int \liminf_j u_j\,d\mu - \int v\,d\mu &= \int \liminf_j (u_j - v)\,d\mu \\ &\leqslant \liminf_j \int (u_j - v)\,d\mu \\ &= \liminf_j \int u_j\,d\mu - \int v\,d\mu\end{aligned}$$

and the claim follows upon subtraction of the *finite (!)* number $\int v\,d\mu$.

(ii) Very similar to (i) by applying Fatou's lemma to the positive, integrable functions $w - u_j \geqslant 0$:

$$\begin{aligned}\int w\,d\mu - \int \limsup_j u_j\,d\mu &= \int \liminf_j (w-u_j)\,d\mu \\ &\leqslant \liminf_j \int (w-u_j)\,d\mu \\ &= \int w\,d\mu - \limsup_j \int u_j\,d\mu\end{aligned}$$

Now subtract the finite number $\int w\,d\mu$ on both sides.

(iii) We had the counterexample, in principle, already in Problem 9.8. Nevertheless...

Consider Lebesgue measure on $\mathbb{R}$. Put $f_j(x) = -\mathbf{1}_{[-2j,-j]}(x)$ and $g_j(x) = \mathbf{1}_{[j,2j]}(x)$. Then $\liminf f_j(x) = 0$ and $\limsup g_j(x) = 0$ for every x and neither admits an integrable minorant resp. majorant.

Problem 10.9 Note the misprint in the statement: the RHS should read $\sum_{j=0}^{\infty} P(|u| \geqslant j)$

We can safely assume that $u \geqslant 0$ (since integrability of u is equivalent to the integrability of $|u|$). Then

$$\begin{aligned}u(x) = \sum_{j=0}^{\infty} u(x)\mathbf{1}_{\{j\leqslant u<j+1\}}(x) &\geqslant \sum_{j=0}^{\infty} j\mathbf{1}_{\{j\leqslant u<j+1\}}(x) \\ &= \sum_{j=0}^{\infty} j\big(\mathbf{1}_{\{j\leqslant u\}}(x) - \mathbf{1}_{\{j+1\leqslant u\}}(x)\big).\end{aligned}$$

Since for fixed x, $u(x) < \infty$, we have $N\mathbf{1}_{\{N+1\leqslant u\}}(x) \xrightarrow{N\to\infty} 0$. Therefore, we can use Abel's summation trick and get

$$\begin{aligned}&\sum_{j=0}^{N} j\big(\mathbf{1}_{\{j\leqslant u\}}(x) - \mathbf{1}_{\{j+1\leqslant u\}}(x)\big) \\ &\quad = 0\cdot\big(\mathbf{1}_{\{0\leqslant u\}}(x) - \mathbf{1}_{\{1\leqslant u\}}(x)\big) + 1\cdot\big(\mathbf{1}_{\{1\leqslant u\}}(x) - \mathbf{1}_{\{2\leqslant u\}}(x)\big) \\ &\qquad + \cdots + N\cdot\big(\mathbf{1}_{\{N\leqslant u\}}(x) - \mathbf{1}_{\{N+1\leqslant u\}}(x)\big) \\ &\quad = \mathbf{1}_{\{1\leqslant u\}}(x) + \mathbf{1}_{\{2\leqslant u\}}(x) + \cdots + \mathbf{1}_{\{N\leqslant u\}}(x) - N\mathbf{1}_{\{N+1\leqslant u\}}(x)\end{aligned}$$

and this proves

$$\sum_{j=0}^{\infty} j\mathbf{1}_{\{j\leqslant u<j+1\}}(x) = \sum_{j=1}^{\infty} \mathbf{1}_{\{j\leqslant u\}}(x).$$

Therefore,

$$u = \sum_{j=0}^{\infty} u\,\mathbf{1}_{\{j\leqslant u<j+1\}} \leqslant \sum_{j=0}^{\infty}(j+1)\mathbf{1}_{\{j\leqslant u<j+1\}}$$
$$\leqslant \sum_{j=0}^{\infty} 2j\mathbf{1}_{\{j\leqslant u<j+1\}}$$
$$= 2\sum_{j=1}^{\infty}\mathbf{1}_{\{j\leqslant u\}}(x) \;\leqslant\; 2u.$$

The claim follows from this, the fact that $\int \text{const.}\,dP = \text{const.}$ and Corollary 9.9:

$$\sum_{j=0}^{\infty} P(\{u\geqslant j\}) = \sum_{j=0}^{\infty}\int \mathbf{1}_{\{u\geqslant j\}}\,dP = \int\sum_{j=0}^{\infty}\mathbf{1}_{\{u\geqslant j\}}\,dP.$$

Problem 10.10 For $u = \mathbf{1}_B$ and $v = \mathbf{1}_C$ we have, because of independence,

$$\int uv\,dP = P(A\cap B) = P(A)P(B) = \int u\,dP\int v\,dP.$$

For positive, simple functions $u = \sum_j \alpha_j \mathbf{1}_{B_j}$ and $v = \sum_k \beta_k \mathbf{1}_{C_k}$ we find

$$\begin{aligned}\int uv\,dP &= \sum_{j,k}\alpha_j\beta_k\int \mathbf{1}_{A_j}\mathbf{1}_{B_k}\,dP\\ &= \sum_{j,k}\alpha_j\beta_k P(A_j\cap B_k)\\ &= \sum_{j,k}\alpha_j\beta_k P(A_j)P(B_k)\\ &= \Big(\sum_j \alpha_j P(A_j)\Big)\Big(\sum_k \beta_k P(B_k)\Big)\\ &= \int u\,dP\int v\,dP.\end{aligned}$$

For measurable $u \in \mathcal{M}^+(\mathcal{B})$ and $v \in \mathcal{M}^+(\mathcal{C})$ we use approximating simple functions $u_k \in \mathcal{E}^+(\mathcal{B})$, $u_k \uparrow u$, and $v_k \in \mathcal{E}^+(\mathcal{C})$, $v_k \uparrow v$. Then, by Beppo Levi,

$$\int uv\,dP = \lim_k \int u_k v_k\,dP = \lim_k \int u_k\,dP \lim_j \int v_j\,dP$$

$$= \int u\,dP \int v\,dP.$$

Integrable independent functions: If $u \in \mathcal{L}^1(\mathcal{B})$ and $v \in \mathcal{L}^1(\mathcal{C})$, the above calculation when applied to $|u|, |v|$ shows that $u \cdot v$ is integrable since

$$\int |uv|\,dP \leqslant \int |u|\,dP \int |v|\,dP < \infty.$$

Considering positive and negative parts finally also gives

$$\int uv\,dP = \int u\,dP \int v\,dP.$$

Counterexample: Just take $u = v$ which are integrable but not square integrable, e.g. $u(x) = v(x) = x^{-1/2}$. Then $\int_{(0,1)} x^{-1/2}\,dx < \infty$ but $\int_{(0,1)} x^{-1}\,dx = \infty$, compare also Problem 10.2.

Problem 10.11 (i) Assume that f^* is $\mathcal{A}^*$-measurable. The problem at hand is to construct $\mathcal{A}$-measurable upper and lower functions g and f. For positive simple functions this is clear: if $f^*(x) = \sum_{j=0}^N \phi_j \mathbf{1}_{B_j^*}(x)$ with $\phi_j \geqslant 0$ and $B_j^* \in \mathcal{A}^*$, then we can use Problem 4.13(v) to find $B_j, C_j \in \mathcal{A}$ with $\mu(C_j \setminus B_j) = 0$

$$B_j \subset B_j^* \subset C_j \implies \phi_j \mathbf{1}_{B_j} \leqslant \phi_j \mathbf{1}_{B_j^*} \leqslant \phi_j \mathbf{1}_{C_j}$$

and summing over $j = 0, 1, \dots, N$ shows that $f \leqslant f^* \leqslant g$ where f, g are the appropriate lower and upper sums which are clearly $\mathcal{A}$ measurable and satisfy

$$\begin{aligned} \mu(\{f \neq g\}) &\leqslant \mu(C_0 \setminus B_0 \cup \dots \cup C_N \setminus B_N) \\ &\leqslant \mu(C_0 \setminus B_0) + \dots + \mu(C_N \setminus B_N) \\ &= 0 + \dots + 0 \;=\; 0. \end{aligned}$$

Moreover, since by Problem 4.13 $\mu(B_j) = \mu(C_j) = \bar{\mu}(B_j^*)$, we have

$$\sum_j \phi_j \mu(B_j) = \sum_j \phi_j \bar{\mu}(B_j^*) = \sum_j \phi_j \mu(C_j)$$

which is the same as

$$\int f\,d\mu = \int f^*\,d\bar{\mu} = \int g\,d\mu.$$

(ii), (iii) Assume that u^* is $\mathcal{A}^*$-measurable; without loss of generality (otherwise consider positive and negative parts) we can assume that $u^* \geqslant 0$. Because of Theorem 8.8 we know that $f_k^* \uparrow u^*$ for $f_k^* \in \mathcal{E}^+(\mathcal{A}^*)$. Now choose the corresponding $\mathcal{A}$-measurable lower and upper functions f_k, g_k constructed in part (i). By considering, if necessary, $\max\{f_1, \dots, f_k\}$ we can assume that the f_k are increasing.

Set $u := \sup_k f_k$ and $v := \liminf_k g_k$. Then $u, v \in \mathcal{M}(\mathcal{A})$, $u \leqslant u^* \leqslant v$, and by Fatou's lemma

$$\begin{aligned}\int v\,d\mu = \int \liminf_k g_k\,d\mu &\leqslant \liminf_k \int g_k\,d\mu \\ &= \liminf_k \int f_k^*\,d\bar\mu \\ &= \int u^*\,d\bar\mu \\ &\leqslant \int v\,d\mu.\end{aligned}$$

Since $f_k \uparrow u$ we get by Beppo Levi and Fatou

$$\begin{aligned}\int u\,d\mu = \sup_k \int f_k\,d\mu &= \liminf_k \int f_k\,d\mu \\ &= \liminf_k \int g_k\,d\mu \\ &\geqslant \int \liminf_k g_k\,d\mu \\ &= \int v\,d\mu \\ &\geqslant \int u\,d\mu\end{aligned}$$

This proves that $\int u\,d\mu = \int v\,d\mu = \int u^*\,d\mu$. This answers part (iii) by considering positive and negative parts.

It remains to show that $\{u \neq v\}$ is a μ-null set. (This does not follow from the above integral equality, cf. Problem 10.16!) Clearly, $\{u \neq v\} = \{u < v\}$, i.e. if $x \in \{u < v\}$ is fixed, we deduce that, for sufficiently large values of k,

$$f_k(x) < g_k(x), \quad k \text{ large}$$

since $u = \sup f_k$ and $v = \liminf_k g_k$. Thus,

$$\{u \neq v\} \subset \bigcup_k \{f_k \neq g_k\}$$

but the RHS is a countable union of μ-null sets, hence a null set itself.

Conversely, assume first that $u \leqslant u^* \leqslant v$ for two $\mathcal{A}$-measurable functions u, v with $u = v$ a.e. We have to show that $\{u^* > \alpha\} \in \mathcal{A}^*$. Using that $u \leqslant u^* \leqslant v$ we find that

$$\{u > \alpha\} \subset \{u^* > \alpha\} \subset \{v > \alpha\}$$

but $\{v > \alpha\}, \{u > \alpha\} \in \mathcal{A}$ and $\{u > \alpha\} \setminus \{v > \alpha\} \subset \{u \neq v\}$ is a μ-null set. Because of Problem 4.13 we conclude that $\{u^* > \alpha\} \in \mathcal{A}^*$.

Problem 10.12 Note the misprint in the statement: for the estimate $\mu_*(E)+\mu_*(F) \leqslant \mu_*(E \uplus F)$ the sets E, F should be disjoint!

Throughout the solution the letters A, B are reserved for sets from $\mathcal{A}$.

(i) a) Let $A \subset E \subset B$. Then $\mu(A) \leqslant \mu(B)$ and going to the $\sup_{A \subset E}$ and $\inf_{E \subset B}$ proves $\mu_*(E) \leqslant \mu^*(E)$.

b) By the definition of μ_* and μ^* we find some $A \subset E$ such that

$$|\mu_*(E) - \mu(A)| \leqslant \epsilon.$$

Since $A^c \supset E^c$ we can enlarge A, if needed, and achieve

$$|\mu^*(E^c) - \mu(A^c)| \leqslant \epsilon.$$

Thus,

$$\begin{aligned}
&|\mu(X) - \mu_*(E) - \mu^*(E^c)| \\
&\qquad \leqslant |\mu_*(E) - \mu(A)| + |\mu^*(E^c) - \mu(A^c)| \\
&\qquad \leqslant 2\epsilon,
\end{aligned}$$

and the claim follows as $\epsilon \to 0$.

c) Let $A \supset E$ and $B \supset F$ be arbitrary majorizing $\mathcal{A}$-sets. Then $A \cup B \supset E \cup F$ and

$$\mu^*(E \cup F) \leqslant \mu(A \cup B) \leqslant \mu(A) + \mu(B).$$

Now we pass on the right-hand side, separately, to the $\inf_{A \supset E}$ and $\inf_{B \supset F}$, and obtain

$$\mu^*(E \cup F) \leqslant \mu^*(E) + \mu^*(F).$$

d) Let $A \subset E$ and $B \subset F$ be arbitrary minorizing $\mathcal{A}$-sets. Then $A \uplus B \subset E \uplus F$ and

$$\mu_*(E \uplus F) \geqslant \mu(A \uplus B) = \mu(A) + \mu(B).$$

Now we pass on the right-hand side, separately, to the $\sup_{A \supset E}$ and $\sup_{B \supset F}$, where we stipulate that $A \cap B = \emptyset$, and obtain

$$\mu_*(E \uplus F) \geqslant \mu_*(E) + \mu_*(F).$$

(ii) By the definition of the infimum/supremum we find sets $A_n \subset E \subset A^n$ such that

$$|\mu_*(A) - \mu(A_n)| + |\mu^*(A) - \mu(A^n)| \leqslant \frac{1}{n}.$$

Without loss of generality we can assume that the A_n increase and that the A^n decrease. Now $A_* := \bigcup_n A_n$, $A^* := \bigcap_n A^n$ are $\mathcal{A}$-sets with $A_* \subset A \subset A^*$. Now, $\mu(A^n) \downarrow \mu(A^*)$ as well as $\mu(A^n) \to \mu^*(E)$ which proves $\mu(A^*) = \mu^*(E)$. Analogously, $\mu(A_n) \uparrow \mu(A_*)$ as well as $\mu(A_n) \to \mu_*(E)$ which proves $\mu(A_*) = \mu_*(E)$.

(iii) In view of Problem 4.13 and (i), (ii), it is clear that

$$\big\{E \subset X \;:\; \mu_*(E) = \mu^*(E)\big\} = \\ \big\{E \subset X \;:\; \exists\, A, B \in \mathcal{A},\; A \subset E \subset B,\; \mu(B \setminus A) = 0\big\}$$

but the latter is the completed σ-algebra $\mathcal{A}^*$. That $\mu^*\big|_{\mathcal{A}^*} = \mu_*\big|_{\mathcal{A}^*} = \bar{\mu}$ is now trivial since μ_* and μ^* coincide on $\mathcal{A}^*$.

Problem 10.13 Let $A \in \mathcal{A}$ and assume that there are non-measurable sets, i.e. $\mathcal{P}(X) \supsetneq \mathcal{A}$. Take some $N \notin \mathcal{A}$ which is a μ-null set. Assume also that $N \cap A = \emptyset$. Then $u = \mathbf{1}_A$ and $w := \mathbf{1}_A + 2 \cdot \mathbf{1}_N$ are a.e. identical, but w is not measurable.

This means that w is only measurable if, e.g. all (subsets of) null sets are measurable, that is if $(X, \mathcal{A}, \mu)$ is complete.

Problem 10.14 The function $\mathbf{1}_{\mathbb{Q}}$ is nowhere continuous but $u = 0$ Lebesgue almost everywhere. That is

$$\{x \;:\; \mathbf{1}_{\mathbb{Q}}(x) \;\text{ is discontinuous}\} = \mathbb{R}$$

while

$$\{x \;:\; \mathbf{1}_{\mathbb{Q}} \neq 0\} = \mathbb{Q} \;\text{ is a Lebesgue null set,}$$

that is $\mathbf{1}_{\mathbb{Q}}$ coincides a.e. with a continuous function but is itself at no point continuous!

The same analysis for $\mathbf{1}_{[0,\infty)}$ yields that

$$\{x \,:\, \mathbf{1}_{[0,\infty)}(x) \ \text{ is discontinuous}\} = \{0\}$$

which is a Lebesgue null set, but $\mathbf{1}_{[0,\infty)}$ cannot coincide a.e. with a continuous function! This, namely, would be of the form $w = 0$ on $(-\infty, -\delta)$ and $w = 1$ on (ϵ, ∞) while it 'interpolates' somehow between 0 and 1 if $-\delta < x < \epsilon$. But this entails that

$$\{x \,:\, w(x) \neq \mathbf{1}_{[0,\infty)}(x)\}$$

cannot be a Lebesgue null set!

Problem 10.15 Let $(A_j)_{j\in\mathbb{N}} \subset \mathcal{A}$ be an exhausting sequence $A_j \uparrow X$ such that $\mu(A_j) < \infty$. Set

$$f(x) := \sum_{j=1}^{\infty} \frac{1}{2^j(\mu(A_j)+1)}\, \mathbf{1}_{A_j}(x).$$

Then f is measurable, $f(x) > 0$ everywhere, and using Beppo Levi's theorem

$$\begin{aligned}\int f\,d\mu &= \int \left(\sum_{j=1}^{\infty} \frac{1}{2^j(\mu(A_j)+1)}\, \mathbf{1}_{A_j}\right) d\mu \\ &= \sum_{j=1}^{\infty} \frac{1}{2^j(\mu(A_j)+1)} \int \mathbf{1}_{A_j}\,d\mu \\ &= \sum_{j=1}^{\infty} \frac{\mu(A_j)}{2^j(\mu(A_j)+1)} \\ &\leqslant \sum_{j=1}^{\infty} 2^{-j} \ = \ 1.\end{aligned}$$

Thus, set $P(A) := \int_A f\,d\mu$. We know from Problem 9.5 that P is indeed a measure.

If $N \in \mathcal{N}_\mu$, then, by Theorem 10.9,

$$P(N) = \int_N f\,d\mu \overset{10.9}{=} 0$$

so that $\mathcal{N}_\mu \subset \mathcal{N}_P$.

Conversely, if $M \in \mathcal{M}_P$, we see that

$$\int_M f \, d\mu = 0$$

but since $f > 0$ everywhere, it follows from Theorem 10.9 that $\mathbf{1}_M \cdot f = 0$ μ-a.e., i.e. $\mu(M) = 0$. Thus, $\mathcal{N}_P \subset \mathcal{N}_\mu$.

Remark. We will see later (cf. Chapter 19, Radon-Nikodým theorem) that $\mathcal{N}_\mu = \mathcal{N}_P$ if and only if $P = f \cdot \mu$ (i.e., if P has a density w.r.t. μ) such that $f > 0$.

Problem 10.16 Well, the hint given in the text should be good enough.