UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Examination in:	MEK-INF3210/4210 — Modeling of fluid flow, heat transfer, and solid deformation.
Day of examination:	Wednesday Dec. 15, 2004
Examination hours:	09.00 – 12.00.

This examination set consists of 7 pages.

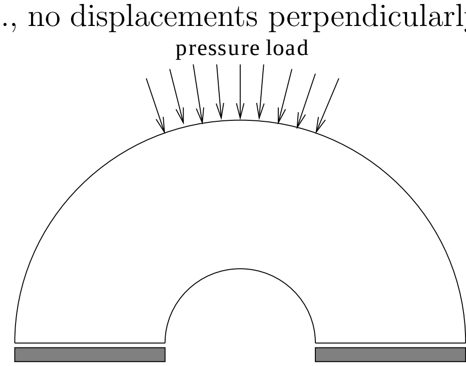
Appendices:	Collection of Mathematical Formulas.
Permitted aids:	Rottmann: Matematische Formelsammlung, one A4 sheet of paper with personal notes

Make sure that your copy of the examination set is complete before you start solving the problems.

Note: each of the points (a), (b), (c), etc., has the same weight.

Exercise 1

An elastic arch as depicted below is subject to a pressure load on a part of its surface. The bottom of the structure is in contact with a rigid, friction-less surface. Assume that the structure is linearly elastic and that we have plain strain conditions (i.e., no displacements perpendicularly to the figure plane).



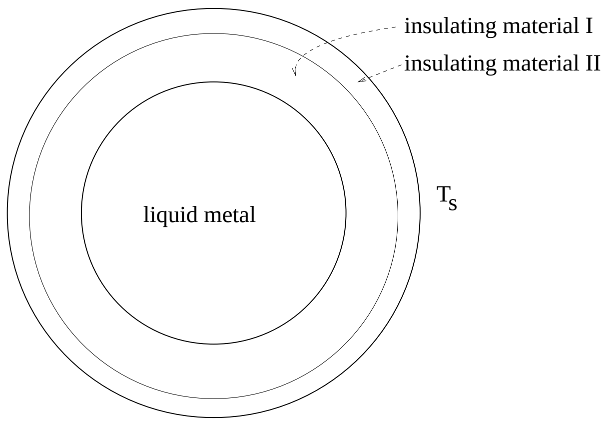
(Continued on page 2.)

(a) We want to compute the von Mises stress as a measure of the stress level in the structure. Set up a mathematical model to accomplish this task (take advantage of symmetry if possible).

(b) Are the stresses computed by the model in (a) valid for any value of the applied pressure load?

Exercise 2

A long, circular, straight pipe for transport of liquid metal is surrounded by two layers of insulating materials.



The metal fills the space $0 \leq r \leq a$, while the insulating materials I and II fill the spaces $a < r \leq b$ and $b < r \leq c$, respectively. A time varying pressure gradient drives the metal flow. Outside the pipe the air temperature is T_s, and this may vary with time, but not in the space direction along the pipe. In a project we want to calculate how the temperature in the insulating materials vary in time and space.

You have just taken over this project from someone else, and in his highly preliminary report you see that the governing equations for heat flow in the fluid and the insulating material is written. You immediately doubt the correctness of the given equations.

(a) In the insulating materials the following equation is given:

$$c_v^{(s)} \frac{\partial T^{(s)}}{\partial t} = \frac{\partial}{\partial r} \left(k^{(s)}(r) \frac{\partial T^{(s)}}{\partial r} \right) \tag{1}$$

(Continued on page 3.)

where

$$k^{(s)}(r) = \begin{cases} k_I, & a < r \leq b, \\ k_{II}, & b < r \leq c \end{cases} \tag{2}$$

Evaluate the correctness of the differential equation and state the physical interpretation of the symbols. Explain also the physical interpretation of each term.

(b) The equations in the liquid metal are listed as follows:

$$\varrho^{(f)} c_v^{(f)} \frac{\partial T^{(f)}}{\partial t} = k^{(f)} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T^{(f)}}{\partial r}\right) + m(T^{(f)})\left(\frac{\partial u}{\partial r}\right)^{n+2}, \tag{3}$$

$$\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(rm(T^{(f)})\left|\frac{\partial u}{\partial r}\right|^{n-1}\frac{\partial u}{\partial r}\right) + g(t)\,. \tag{4}$$

Again, you are asked to evaluate whether these equations are correct. The most important issue (on this exam) is to *point out errors*. A complete derivation of the correct equations, such that each error is explained in detail, is not necessary (but it might be a safe way of anwsering the exercise if you do not remember how the different terms should be).

(c) In the preliminary report you find the following boundary conditions:

$$\frac{\partial u}{\partial r} = 0, \quad r = 0, \tag{5}$$

$$\frac{\partial T^{(f)}}{\partial r} = 0, \quad r = 0, \tag{6}$$

$$u = 0, \quad r = a, \tag{7}$$

$$\frac{\partial T^{(f)}}{\partial r} - \frac{\partial T^{(s)}}{\partial r} = 0, \quad r = a, \tag{8}$$

$$T^{(f)} = T^{(s)}, \quad r = a, \tag{9}$$

$$-k^I \frac{\partial T^{(s)}}{\partial r} = -k_{II}\frac{\partial T^{(s)}}{\partial r}, \quad r = b, \tag{10}$$

$$T^{(s)} = T^{(s)}, \quad r = b, \tag{11}$$

$$k^{II}\frac{\partial T^{(s)}}{\partial r} + h_T(T^{(s)} - T_s) = 0, \quad r = c\,. \tag{12}$$

Set up the correct boundary conditions that you need if you choose the correct version of (1), (3) and (4) as the governing differential equations.

(Continued on page 4.)

Exercise 3

(a) Find at least two physical interpretations of the boundary-value problem

$$u'' = 0, \quad u(0) = 0,\ u(1) = 1\,.$$

In each case, state the physical meaning of u, the boundary conditions, and the geometry (i.e., what the interval $[0, 1]$ corresponds to in nature). You do not need to derive the boundary-value problem from more general equations of continuum mechanics, but you should mention which of the general three-dimensional equations that constitute the starting point for deriving this simplified model.

Collection of Mathematical Formulas

The Operator in Navier's Equation with Radial Symmetry. Let r be the radial coordinate in cylindrical coordinates. If

$$\mathcal{L}(\boldsymbol{u}) = (\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) + \mu\nabla^2\boldsymbol{u}, \tag{13}$$

then

$$\mathcal{L}(u(r)\boldsymbol{i}_r) = \boldsymbol{i}_r(\lambda + 2\mu)\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(ru)\right)\,. \tag{14}$$

Moreover,

$$\mathcal{L}(u(r)\boldsymbol{i}_r + w(z)\boldsymbol{k}) = \boldsymbol{i}_r(\lambda + 2\mu)\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(ru)\right) + \boldsymbol{k}(\lambda + 2\mu)w''(z)\,. \tag{15}$$

In spherical coordinates we have

$$\mathcal{L}(u(r)\boldsymbol{i}_r) = \boldsymbol{i}_r(\lambda + 2\mu)\frac{d}{dr}\left(\frac{1}{r^2}\frac{d}{dr}(r^2u)\right) \tag{16}$$

Strains with Radial Symmetry. Radial displacement, $\boldsymbol{u} = u(r)\boldsymbol{i}_r$, in cylindrical coordinates (r, θ, z) have corresponding (infinitesimal) strains

$$E = \frac{du}{dr}\boldsymbol{i}_r\boldsymbol{i}_r + \frac{u}{r}\boldsymbol{i}_\theta\boldsymbol{i}_\theta \tag{17}$$

(Continued on page 5.)

or

$$\varepsilon_{rr} = \frac{du}{dr}, \quad \varepsilon_{\theta\theta} = \frac{u}{r}\,. \tag{18}$$

In spherical coordinates (r, θ, ϕ) the strains corresponding to a deformation $\boldsymbol{u} = u(r)\boldsymbol{i}_r$ become

$$E = \frac{du}{dr}\boldsymbol{i}_r\boldsymbol{i}_r + \frac{u}{r}(\boldsymbol{i}_\theta\boldsymbol{i}_\theta + \boldsymbol{i}_\phi\boldsymbol{i}_\phi) \tag{19}$$

or

$$\varepsilon_{rr} = \frac{du}{dr}, \quad \varepsilon_{\theta\theta} = \varepsilon_{\phi\phi} = \frac{u}{r}\,. \tag{20}$$

The Divergence of the Stress Tensor for Radial Symmetry. If the deformation is radial such that the only non-vanishing stress tensor components in cylindrical coordinates are σ_{rr} and $\sigma_{\theta\theta}$, the divergence of the stress tensor becomes

$$\boldsymbol{i}_r \cdot \nabla \cdot \{\sigma_{ij}\} = \frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r}\,. \tag{21}$$

The corresponding expression in spherical coordinates has the form

$$\boldsymbol{i}_r \cdot \nabla \cdot \{\sigma_{ij}\} = \frac{d\sigma_{rr}}{dr} + 2\frac{\sigma_{rr} - \sigma_{\theta\theta}}{r}\,. \tag{22}$$

The Compatibility Equation for Radial Symmetry.. The compatibility equation in cylindrical coordinates, both for $\boldsymbol{u} = u(r)\boldsymbol{i}_r$ (plane strain) and $\boldsymbol{u} = u(r)\boldsymbol{i}_r + w(z)\boldsymbol{k}$ (e.g. plane stress), takes the form

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\left(\sigma_{rr} + \sigma_{\theta\theta}\right)\right) = 0\,. \tag{23}$$

Invariants. The invariants I_B, II_B og III_B of a tensor B_{ij} are given by the expressions

$$I_B = B_{ii} = \mathrm{tr}B, \tag{24}$$

$$II_B = \frac{1}{2}\left(B_{ii}B_{jj} - B_{ij}B_{ji}\right), \tag{25}$$

$$III_B = \det\{B_{ij}\} = \frac{1}{2}\left(B_{ii}B_{jj}B_{kk} - 3B_{ii}B_{jk}B_{kj} + 2B_{ij}B_{jk}B_{ki}\right)\,. \tag{26}$$

Hooke's Generalized Law. Using the Lamé constants λ og μ one can write Hooke's generalized law for an isotropic linearly elastic material on the form

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}\,. \tag{27}$$

With Young's modulus E og Poisson's ratio ν the law can be written

$$\varepsilon_{ij} = -\frac{\nu}{E}\sigma_{kk}\delta_{ij} + \frac{1+\nu}{E}\sigma_{ij}\,. \tag{28}$$

(Continued on page 6.)

Yield Criteria. Tresca's yield criterion can be written

$$2\tau_m = Y \tag{29}$$

where τ_m denotes maximum shear stress, whereas von Mises' yield criterion can be compactly expressed as

$$\sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}} = Y \tag{30}$$

with the stress deviator tensor being defined as $\sigma'_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$. The parameter Y can be interpreted as the yield stress in uni-axial tension. Written out with the original stress tensor components, von Mises' criterion reads

$$\left(\frac{1}{2}\left[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2\right] + 3\left[\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2\right]\right)^{\frac{1}{2}} = Y \tag{31}$$

Derivatives of Unit Vectors in Cylindrical and Spherical Coordinates. Let $\boldsymbol{i}_r$, $\boldsymbol{i}_\theta$ og $\boldsymbol{i}_\phi$ be unit vectors in r-, ϕ- and θ-direction in spherical coordinates (r, θ, ϕ), and

$$\begin{aligned} x &= r\cos\theta\sin\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\phi\,. \end{aligned}$$

Then

$$\nabla = \boldsymbol{i}_r\frac{\partial}{\partial r} + \boldsymbol{i}_\phi\frac{1}{r}\frac{\partial}{\partial\phi} + \boldsymbol{i}_\theta\frac{1}{r\sin\phi}\frac{\partial}{\partial\theta} \tag{32}$$

and

$$\frac{\partial\boldsymbol{i}_r}{\partial\phi} = \boldsymbol{i}_\phi,\quad \frac{\partial\boldsymbol{i}_r}{\partial\theta} = \sin\phi\,\boldsymbol{i}_\theta \tag{33}$$

The Laplace Operator with Radial Symmetry. The Laplace operator has the following forms in cylindrical and spherical coordinates (r is the radial coordinate and $k(r)$ is some function):

$$\nabla\cdot(k(r)\nabla u(r)) = \frac{1}{r}\frac{d}{dr}\left(rk(r)\frac{du}{dr}\right) \text{ cylindrical coord.,} \tag{34}$$

$$\nabla\cdot(k(r)\nabla u(r)) = \frac{1}{r^2}\frac{d}{dr}\left(r^2k(r)\frac{du}{dr}\right) \text{ spherical coord..} \tag{35}$$

(Continued on page 7.)

Stresses in a Sphere Subject to Inner and Outer Pressure. A thick-walled sphere is subject to an inner pressure p_i at the boundary $r = a$ and an outer pressure p_o at the boundary $r = b$. The non-vanishing stress tensor components then becomes

$$\sigma_{rr} = \frac{p_o b^3(r^3 - a^3)}{r^3(a^3 - b^3)} + \frac{p_i a^3(b^3 - r^3)}{r^3(a^3 - b^3)}, \tag{36}$$

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{p_o b^3(2r^3 + a^3)}{2r^3(a^3 - b^3)} - \frac{p_i a^3(2r^3 + b^3)}{2r^3(a^3 - b^3)}. \tag{37}$$

Axi-Symmetric Rotation. For a vector field $\boldsymbol{v} = u(r)\boldsymbol{i}_\theta$ (e.g. $\boldsymbol{v}$ being velocity or displacement) in cylindrical coordinates we have that

$$\frac{1}{2}(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T) = \frac{1}{2}\left(\frac{du}{dr} - \frac{u}{r}\right)(\boldsymbol{i}_\theta\boldsymbol{i}_r + \boldsymbol{i}_r\boldsymbol{i}_\theta). \tag{38}$$

The divergence of the corresponding stress tensor becomes

$$\nabla \cdot \{\sigma_{ij}\} = \left(\frac{\partial\sigma_{r\theta}}{\partial r} + 2\frac{\sigma_{r\theta}}{r}\right)\boldsymbol{i}_\theta, \tag{39}$$

provided that the only non-vanishing stress tensor component is $\sigma_{r\theta} = \sigma_{\theta r}$.

END