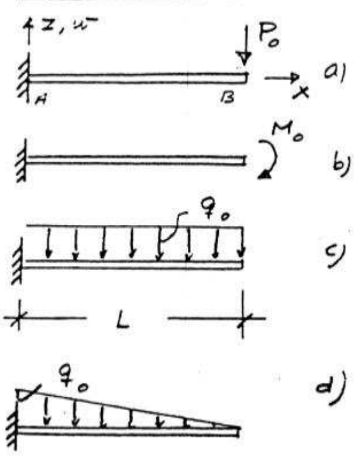
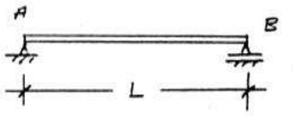
OPPGAVE 1a



Beregn forskyvning w_B og rotasjon (vinkelendring) θ_B ved den frie enden for uttragerbjelkene i Fig. a), b), c) og d).

Bjelkene har konstant bøyestivhet EI. Skjær deformasjoner kan neglisjeres.

OPPGAVE 2a

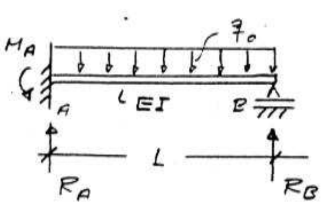


Bøyelinjens ligning for en fritt opplagt bjelke er

$$w = \frac{q_o x}{360\, L \cdot EI}\left(7L^4 - 10L^2x^2 + 3x^4\right)$$

Hvordan er belastningen?

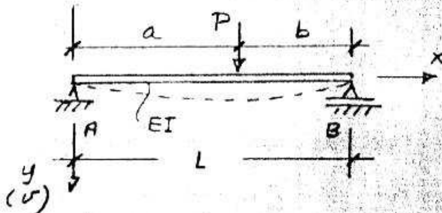
OPPGAVE 3a



a) Beregn bøyelinje og lagerreaksjoner (R_A, R_B og M_A) for den en gang statisk ubestemte bjelken i figuren. Skjærdef. neglisjeres.

b) Tegn opp bøyelinje, og moment- og skjærkraft-diagram.

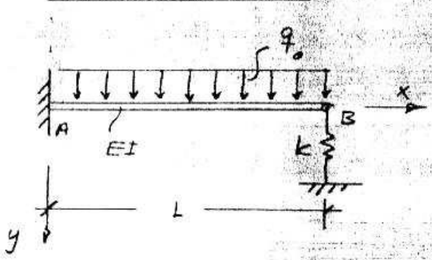
OPPGAVE .1b



a) Bestemm bøyelinjen for den fritt opplagte bjelken med konsentrert last P som vist i fig.. Bjelken har konstant bøyestivhet EI

UTGÅR { b) Bestemm også skjærdeformasjonen av bjelken som har skjærstivhet GA/μ, og tegn opp skjærforskyvningen langs bjelken.

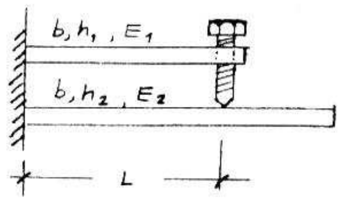
OPPGAVE .2b



Bjelken i figuren, belastet med en jevnt fordelt last q_o i negativ z-retning, er fast innspennt ved A og elastisk opplagret på en vertikal translasjonsfjær med fjærstivhet k ved B.

Bestemm bøyelinjen og lagerreaksjoner ved A og B, og diskuter grensetilfellene som fås når $k \to 0$ og $k \to \infty$.

OPPGAVE .3b



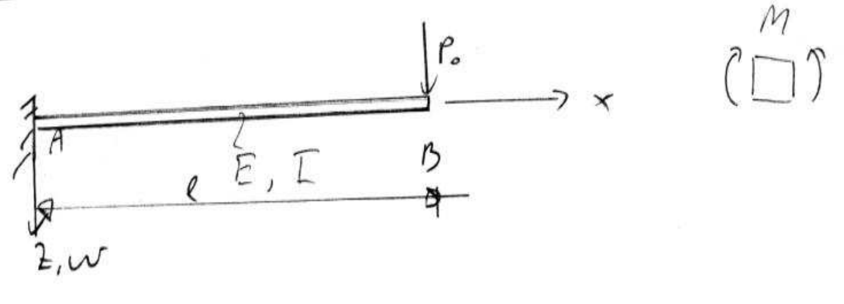
To bjelker med rektangulære tverrsnitt (bredde b, høyde hhv. h_1 og h_2) og elastisitetsmodul E_1 hhv. E_2 er fast innspennt i en vegg. En skrue, gjenget i den øvre bjelken i avstand L fra veggen, skrues til kontakt med den nedre bjelken og deretter videre et stykke δ.

Beregn maksimal spenning i den øvre bjelken og vis at den kan skrives $\sigma = 3\delta h_1 h_2^3 E_1 E_2 / 2L^2(E_1 h_1^3 + E_2 h_2^3)$.

Oppgavesett 7 a)

Opps. 7

c)



$$M(x) = P_0 (x - l)$$

Diff. likn:

$$w'' = \frac{-M}{EI}$$

$$w'' = \frac{P_0}{EI} (l - x)$$

$$w' = \frac{P_0}{EI} \left(l x - \frac{1}{2} x^2 \right) + C_1$$

$$w = \frac{P_0}{EI} \left(\frac{1}{2} l x^2 - \frac{1}{6} x^3 \right) + C_1 x + C_2$$

Randbetingelser:

$$w'(0) = w(0) = 0$$

$$w(0) = 0 \Rightarrow C_2 = 0$$

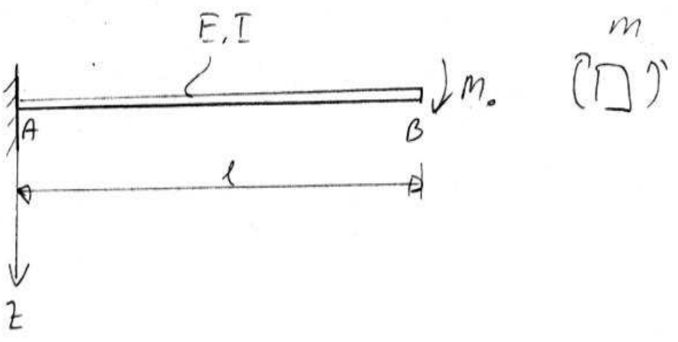
$$w'(0) = 0 \Rightarrow C_1 = 0$$

$$\underline{w(x) = \frac{P_0}{6EI}\left(3\ell x^2 - x^3\right)}$$

$$w_B = w(\ell) = \frac{P_0}{6EI}\left(3\ell\,\ell^2 - \ell^3\right) = \underline{\frac{P_0 \ell^3}{3\,EI}}$$

$$\theta_B = w'(\ell) = \frac{P_0}{EI}\left(\ell \cdot \ell - \tfrac{1}{2}\ell^2\right) = \underline{\frac{P_0 \ell^2}{2EI}}$$

b)



$$w'' = -\frac{M}{EI}$$

$M = -M_0$

$$w'' = -\frac{M_0}{EI}$$

$$w' = -\frac{M_0}{EI}x + C_1$$

$$w = -\frac{M_0}{2EI}x^2 + C_1 x + C_2$$

<u>Randbetingelser:</u>

$w'(0) = w(0) = 0$

$w'(0) = 0 \Rightarrow C_1 = 0$

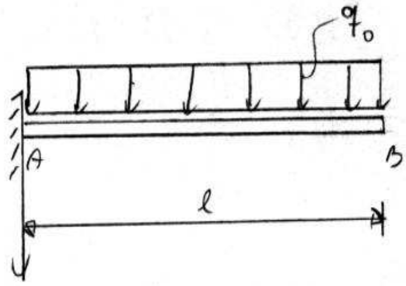
$w(0) = 0 \Rightarrow C_2 = 0$

$$w(x) = -\frac{M_0}{2EI} x^2$$

$$w_B = w(l) = \frac{M_0 l^2}{2EI}$$

$$\theta_B = w'(l) = \frac{M_0 l}{EI}$$

c)



$$w'' = -\frac{M}{EI} \qquad (1)$$

$$\frac{d}{dx}(1) \Rightarrow w''' = -\frac{1}{EI}\frac{dM}{dx}$$ (Konstant EI)

$$\frac{dM}{dx} = Q \Rightarrow w''' = -\frac{Q}{EI} \qquad (2)$$

$$\frac{d}{dx}(2) \Rightarrow w^{(4)} = -\frac{1}{EI}\frac{dQ}{dx}$$

$$\frac{dQ}{dx} = -q \Rightarrow \boxed{w^{(4)} = \frac{q}{EI}} \qquad (3)$$ (Gjelder for bjelker med konstant stivhet)

Benytter denne i det videre

Får:

$$w^{(4)} = \frac{q_0}{EI}$$

$$w''' = \frac{q_0}{EI}x + C_1$$

$$w'' = \frac{q_0}{2EI}x^2 + C_1 x + C_2$$

$$w' = \frac{q_0}{6EI}x^3 + \frac{C_1}{2}x^2 + C_2 x + C_3$$

$$w = \frac{q_0}{24EI}x^4 + \frac{C_1}{6}x^3 + \frac{C_2}{2}x^2 + C_3 x + C_4$$

Randbetingelser

$$\left.\begin{array}{ll} w'''(l) = 0 & \text{(Skjærkraft)} \\ w''(l) = 0 & \text{(Moment)} \end{array}\right\} \text{Mekaniske}$$

$$\left.\begin{array}{l} w'(0) = 0 \\ w(0) = 0 \end{array}\right\} \text{Kinematiske.}$$

$$w'''(l) = 0$$

$$\Rightarrow \frac{q_0}{EI}l + C_1 = 0 \Rightarrow C_1 = -\frac{q_0}{EI}l$$

$$w'''(x) = \frac{q_0}{EI}(x - l)$$

$$w''(l) = 0$$

$$\Rightarrow \frac{q_0}{2EI}l^2 - \frac{q_0}{EI}l^2 + C_2 = 0$$

$$\Rightarrow C_2 = \frac{q_0}{2EI}l^2$$

$$w'(0)=0 \Rightarrow C_3 = 0$$

$$w(0)=0 \Rightarrow C_4 = 0$$

$$w(x) = \frac{q_0}{24EI}x^4 - \frac{q_0 l}{6EI}x^3 + \frac{q_0 l^2}{4EI}x^2$$

$$= \frac{q_0}{24EI}\left(x^4 - 4 l x^3 + 6 l^2 x^2\right)$$

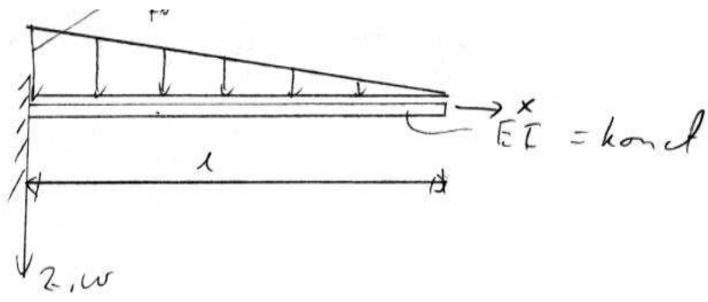
$$w_B = w(l) = \frac{q_0}{24EI}\left(l^4 - 4 l^4 + 6 l^4\right)$$

$$= \frac{3q_0 l^4}{24EI} = \underline{\frac{q_0 l^4}{8EI}}$$

$$\theta_B = w'(l) = \frac{q_0}{6EI}l^3 - \frac{q_0}{2EI}l^3 + \frac{q_0}{2EI}l^3$$

$$= \underline{\frac{q_0 l^3}{6EI}}$$

d)



$$q = q_0\left(1 - \frac{x}{l}\right)$$

$$w^{(4)} = \frac{q}{EI}$$

$$w''' = \frac{q_0}{EI}\left(x - \frac{x^2}{2l}\right) + C_1$$

$$w'' = \frac{q_0}{EI}\left(\frac{1}{2}x^2 - \frac{x^3}{6l}\right) + C_1 x + C_2$$

$$w' = \frac{q_0}{EI}\left(\frac{1}{6}x^3 - \frac{x^4}{24l}\right) + \frac{C_1}{2}x^2 + C_2 x + C_3$$

$$w = \frac{q_0}{EI}\left(\frac{1}{24}x^4 - \frac{x^5}{120l}\right) + \frac{C_1}{6}x^3 + \frac{C_2}{2}x^2 + C_3$$

Randbetingelser:

$w'''(l) = 0$

$w''(l) = 0$

$w'(0) = 0$

$w(0) = 0$

$$w'''(l) = 0$$

$$\Rightarrow \frac{q_0}{EI}\left(l - \frac{l^2}{2l}\right) + C_1 = 0$$

$$C_1 = -\frac{q_0 l}{2EI}$$

$$w''(l) = 0$$

$$\Rightarrow \frac{q_0}{EI}\left(\frac{1}{2}l^2 - \frac{l^3}{6l}\right) - \frac{q_0 l^2}{2EI} + C_2 = 0$$

$$\frac{q_0 l^2}{3EI} - \frac{q_0 l^2}{2EI} + C_2 = 0$$

$$C_2 = \frac{q_0 l^2}{6EI}$$

$$w'(0) = 0 \Rightarrow C_3 = 0$$

$$w(0) = 0 \Rightarrow C_4 = 0$$

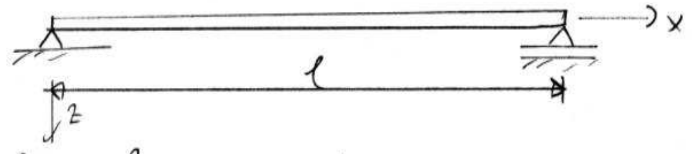
$$w(x) = \frac{q_0}{120EIl}\left(5x^4 l - x^5\right) - \frac{q_0 l}{12EI}x^3 + \frac{q_0 l^2}{12EI}x^2$$

$$w_B = w(l) = \underline{\underline{\frac{q_0 l^4}{30EI}}}$$

$$\theta_B = w'(l) = \frac{q_0}{EI}\left(\frac{1}{6}l^3 - \frac{l^3}{24}\right) - \frac{q_0 l^3}{4EI} + \frac{q_0 l^3}{6EI} \cdot$$

$$\theta_B = \frac{q_0 l^3}{EI}\left(\frac{1}{6} - \frac{1}{24} - \frac{1}{4} + \frac{1}{6}\right) = \frac{5 q_0 l^3}{24EI}$$

Opps. 2



Bøyelinjens likning sitt som:

$$w = \frac{q_0 x}{360 l EI}\left(7l^4 - 10l^2x^2 + 3x^4\right)$$

Diff-likning ved konstant stivhet:

$$\boxed{w^{(4)} = \frac{q}{EI}} \Rightarrow \quad q = EIw^{(4)}$$

$$w' = \frac{q_0}{360 l EI}\left(7l^4 - 30l^2x^2 + 15x^4\right)$$

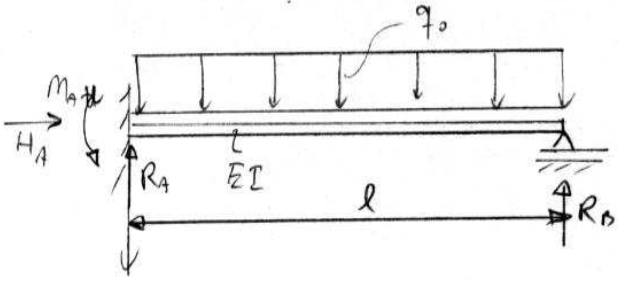
$$w'' = \frac{q_0}{360 l EI}\left(-60l^2x + 60x^3\right)$$

$$w''' = \frac{q_0}{360 EI l}\left(-60l^2 + 180x^2\right)$$

$$w^{(4)} = \frac{q_0}{360 EI l} \cdot 360x = \frac{q_0 x}{EI l}$$

$$\Rightarrow \quad q = \frac{EI q_0 x}{EI l} = \underline{q_0 \frac{x}{l}}$$

Opps. 3



$$\Sigma F_x = 0 \Rightarrow H_A = 0$$

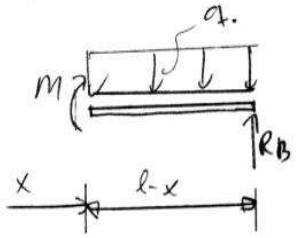
$$\Sigma M_A = 0 \Rightarrow -M_A + \frac{q_0 l^2}{2} - R_B \cdot l = 0$$

$$\Rightarrow -M_A = R_B\, l - \frac{q_0 l^2}{2} \qquad (1)$$

$$\Sigma F_y = 0 \Rightarrow R_A + R_B - q_0 l = 0 \qquad (2)$$

Diff-likning:

$$w'' = -\frac{M}{EI} \qquad (3)$$



$$\Sigma M_s = 0 \Rightarrow M + q_0 \frac{(l-x)(l-x)}{2} - R_B (l-x) = 0$$

$$\Rightarrow M = R_B (l-x) - \frac{q_0}{2}\left(l^2 - 2lx + x^2\right)$$

Inn i (3)

$$w'' = \frac{q_0}{2EI}\left(l^2 - 2lx + x^2\right) + \frac{R_B}{EI}(x-l)$$

$$w' = \frac{q_0}{2EI}\left(l^2x - lx^2 + \frac{1}{3}x^3\right) + \frac{R_B}{EI}\left(\frac{x^2}{2} - lx\right) + C_1$$

$$w = \frac{q_0}{2EI}\left(\frac{l^2x^2}{2} - \frac{lx^3}{3} + \frac{x^4}{12}\right) + \frac{R_B}{EI}\left(\frac{x^3}{6} - \frac{lx^2}{2}\right) + C_1x + C_2$$

<u>Randbetingelser:</u>

$$w(0) = w'(0) = 0$$

$$\Rightarrow C_1 = C_2 = 0$$

Kinematisk betingelse for bestemmelse av R_B:

$$w(l) = 0$$

$$\Rightarrow \frac{q_0}{2\cancel{EI}}\left(\frac{l^4}{2} - \frac{l^4}{3} + \frac{l^4}{12}\right) + \frac{R_B}{\cancel{EI}}\left(\frac{l^3}{6} - \frac{l^3}{2}\right) = 0$$

$$\frac{3q_0 l^4}{24\,EI} - \frac{2R_B l^3}{6} = 0$$

$$\underline{R_B = \frac{3q_0 l}{8}}$$

$16 \cdot \frac{3}{2}$

$$w(x) = \frac{q_0}{24EI}\left(6l^2x^2 - 4lx^3 + x^4\right) + \frac{3q_0 l}{8EI}\left(\frac{x^3}{6} - \frac{lx^2}{2}\right)$$

$$= \frac{q_0}{48EI}\left(12l^2x^2 - 8lx^3 + 2x^4 + 3lx^3 - 3l^2x^2\right)$$

$$= \underline{\frac{q_0}{48EI}\left(3l^2x^2 - 5lx^3 + 2x^4\right)}$$

(1) sin:

$$-M_A = \frac{3q_0 l^2}{8} - \frac{q_0 l^2}{2} = \underline{-\frac{q_0 l^2}{8}}$$

(2) sin:

$$R_A = q_0 l - \frac{3q_0 l}{8} = \underline{\frac{5q_0 l}{8}}$$

b)

$$M = \frac{3q_0 l}{8}(l-x) - \frac{q_0}{2}(l^2 - 2lx + x^2)$$

$$= -\frac{q_0}{8}\left(l^2 - 5lx + 4x^2\right)$$

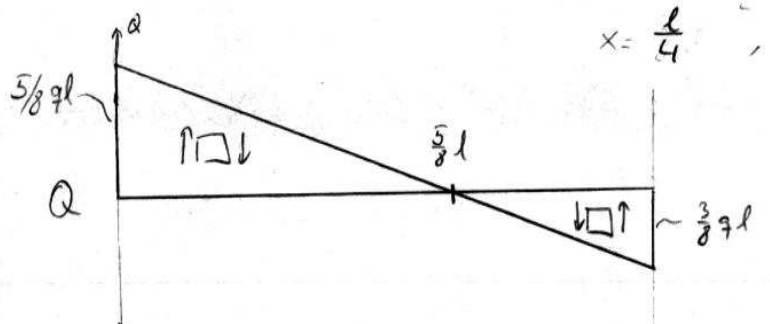
$$x^2 - \frac{5}{4}lx + \frac{l^2}{4} = 0$$

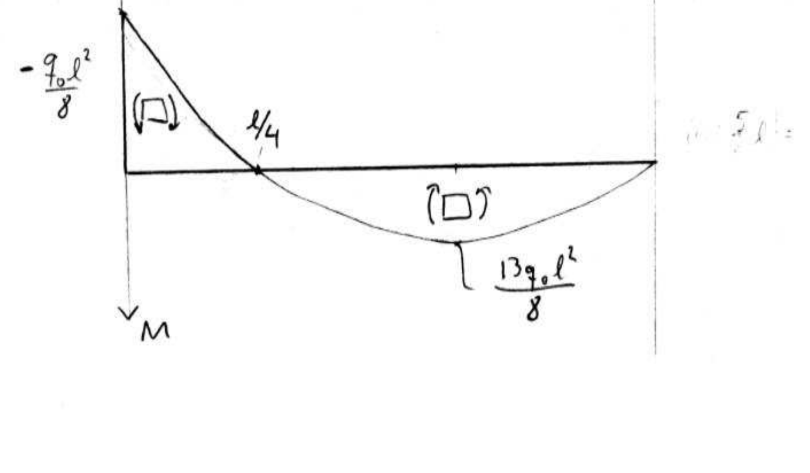
$$x = \frac{\frac{5}{4}l \pm \sqrt{\frac{25}{16}l^2 - 4\cdot 1 \cdot \frac{l^2}{4}}}{2}$$

$$x = \frac{5}{8}l \pm \frac{3}{8}l$$

$$x = \frac{l}{4}, \quad x = l$$

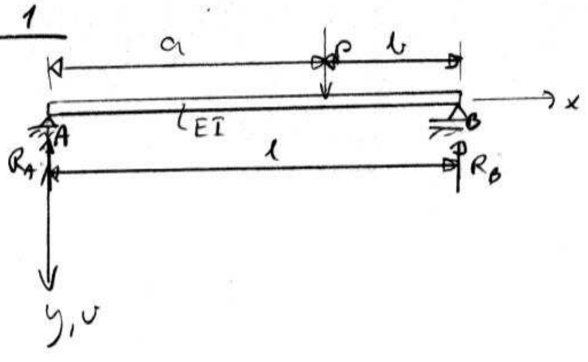
$$Q = \frac{dM}{dx} = -\frac{q_0}{8}(5l - 8x)$$





Oppgavesett 9 b)

Oppg. 1

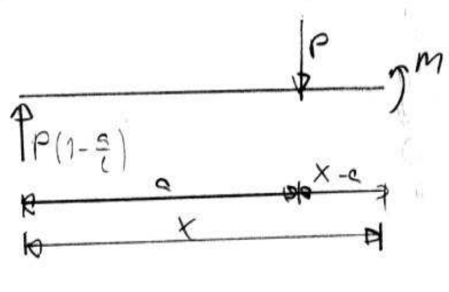


Opplagerkrefter

$$\Sigma M_A = 0 \Rightarrow \quad Pa - R_B \cdot l = 0 \Rightarrow \underline{R_B = \frac{P \cdot a}{l}}$$

$$\Sigma F_y = 0 \Rightarrow \quad R_A + R_B - P = 0 \Rightarrow \underline{R_A = P\left(1 - \frac{a}{l}\right)}$$

Snittkrefter:



$$[x-a] = \begin{cases} 0 & x < a \\ (x-a) & x > a \end{cases}$$

$$M + P[x-a] - P\left(1 - \frac{a}{l}\right) \cdot x = 0$$

$$M = P\left(1 - \frac{a}{l}\right)x - P[x-a]$$

Diff-likning:

$$v'' = -\frac{M}{EI}$$

$$v'' = \frac{1}{EI}\left(P[x-a] - P\left(1-\frac{a}{l}\right)x\right)$$

$$v' = \frac{P}{EI}\left(\frac{1}{2}[x-a]^2 - \frac{1}{2}\left(1-\frac{a}{l}\right)x^2\right) + C_1$$

$$v = \frac{P}{2EI}\left(\frac{2}{3}[x-a]^3 - \frac{1}{3}\left(1-\frac{a}{l}\right)x^3\right) + C_1 x + C_2$$

$$= \frac{P}{6EI}\left([x-a]^3 - \left(1-\frac{a}{l}\right)x^3\right) + C_1 x + C_2$$

Randbetingelser:

$$(l-a)(l^2-2la+a^2)$$
$$= l^3 - 2l^2a + a^2l - al^2 + 2la^2 - a^3$$
$$= l^3 - 3l^2a + 3a^2l - a^3$$

$v(0) = 0$

$v(l) = 0$

$$v(0) = 0 \Rightarrow C_2 = 0$$

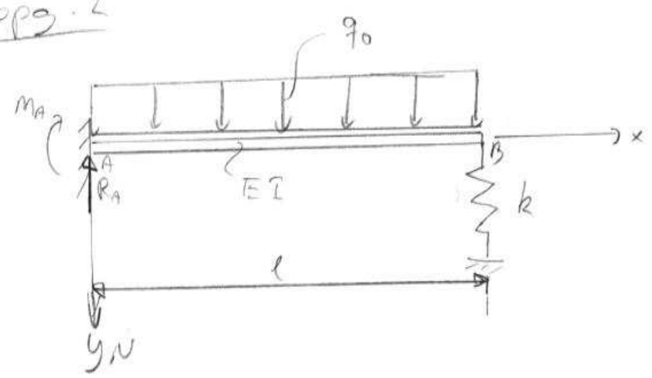
$$v(l) = 0 \Rightarrow \frac{P}{6EI}\left((l-a)^3 - \left(1-\frac{a}{l}\right)l^3\right) + C_1 l = 0$$

$$C_1 = -\frac{P}{6EIl}\left(\cancel{l^3} - 3l^2a + 3a^2l - a^3 - \cancel{l^3} + al^2\right)$$

$$= \frac{P}{6EI}\left(\frac{a^3}{l} + 2la - 3a^2\right)$$

$$\Rightarrow \quad \underline{v = \frac{P}{6EI}\left([x-a]^3 - \left(1-\frac{a}{l}\right)x^3 + \left(\frac{a^3}{l} + 2la - 3a^2\right)x\right)}$$

Opps. 2

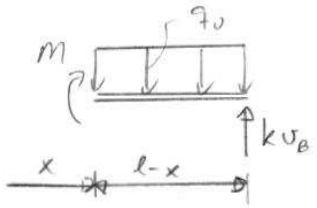


$$\Sigma F_y = 0 \Rightarrow R_A - q_0 l + k v_B = 0 \qquad (1)$$

$$\Sigma M_A = 0 \Rightarrow M_A + \frac{q_0 l^2}{2} - k v_B l = 0 \qquad (2)$$

Diff. likning:

$$v'' = -\frac{M}{EI}$$



$$\Sigma M_1 = 0 \Rightarrow M + \frac{q_0 (l-x)^2}{2} - k v_B (l-x) = 0$$

$$\Rightarrow M = k v_B (l-x) - \frac{q_0}{2}(l-x)^2$$

$$\Rightarrow v'' = \frac{1}{EI}\left(\frac{q_0}{2}(l-x)^2 - k v_B (l-x)\right)$$

$$v' = \frac{1}{EI}\left(\frac{q_0}{2}\left(l^2x - lx^2 + \frac{1}{3}x^3\right) - k v_B\left(lx - \frac{1}{2}x^2\right)\right) + C_1$$

$$v = \frac{1}{EI}\left(\frac{q_0}{2}\left(\frac{1}{2}l^2x^2 - \frac{1}{3}lx^3 + \frac{1}{12}x^4\right) - k v_B\left(\frac{1}{2}lx^2 - \frac{1}{6}x^3\right)\right) + C_1x + C_2$$

Randbedingungen:

$$v'(0) = v(0) = 0 \Rightarrow C_1 = C_2 = 0$$

$$v_B = v(l) = \frac{1}{EI}\left(\frac{q_0}{24}\left(6l^4 - 4l^4 + l^4\right) - \frac{k}{6}v_B\left(3l^3 - l^3\right)\right)$$

$$v_B\left(1 + \frac{kl^3}{3EI}\right) = \frac{q_0l^4}{8EI}$$

$$v_B = \frac{3q_0l^4}{8(3EI + kl^3)} \qquad (3)$$

$$\Rightarrow v = \frac{1}{EI}\left(\frac{q_0}{24}\left(6l^2x^2 - 4lx^3 + x^4\right) - \frac{kq_0l^4}{16(3EI + kl^3)}\left(3lx^2 - x^3\right)\right)$$

$$= \frac{q_0}{48EI}\left(12l^2x^2 - 8lx^3 + 2x^4 - \frac{3kl^4}{(3EI + kl^3)}\left(3lx^2 - x^3\right)\right) \qquad (4)$$

(3) inn i (1) og (2) gir opplagerkrefter:

$$R_A - q_0 l + \frac{3 q_0 k l^4}{8(3EI + kl^3)} = 0$$

$$R_A = q_0 l \left(1 - \frac{3kl^3}{8(3EI+kl^3)}\right)$$

$$M_A + \frac{q_0 l^2}{2} - \frac{3 q_0 k l^5}{8(3EI+kl^3)} = 0$$

$$M_A = \frac{q_0 l^2}{2}\left(\frac{3kl^3}{4(3EI+kl^3)} - 1\right)$$

Betrakter grensetilfellene $k \to 0$ og $k \to \infty$

$\underline{k \to 0}$

$$\lim_{k\to 0} R_A = \lim_{k\to 0} q_0 l \left(1 - \frac{3kl^3}{8(3EI+kl^3)}\right)$$

$$= q_0 l - \lim_{k \to 0} \frac{3kl^3}{8(3EI+kl^3)}$$

$$= \underline{q_0 l}$$

$$\lim_{k\to 0} M_A = \lim_{k\to 0} \frac{q_0 l^2}{2}\left(\frac{3kl^3}{4(3EI+kl^3)} - 1\right) = -\frac{q_0 l^2}{2}$$

$$\lim_{k\to 0} v = \lim_{k\to 0} \frac{q_0}{48EI}\left(12l^2x^2 - 8lx^3 + 2x^4 - \frac{3q_0 kl^4}{(3EI+kl^3)}(3lx^2 - x^3)\right)$$

$$= \frac{q_0}{24EI}\left(6l^2x^2 - 4lx^3 + x^4\right)$$

Stemmer med opps 9a 1c)

$k \to \infty$

$$\lim_{k\to\infty} R_A = \lim_{k\to\infty} q_0 l\left(1 - \frac{3kl^3}{8(3EI+kl^3)}\right)$$

$$= q_0 l - \lim_{k\to\infty} \frac{3kl^3 q_0 l}{8(3EI+kl^3)}$$

$$= q_0 l - \lim_{k\to\infty} \frac{3l^3 q_0 l}{8\left(\frac{3EI}{k}+l^3\right)} = q_0 l - \frac{3}{8}q_0 l = \frac{5}{8}q_0 l$$

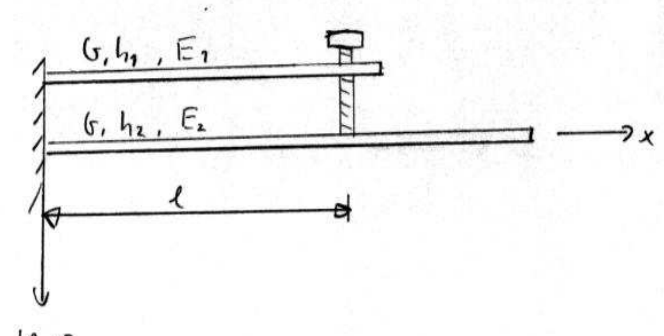
$$\lim_{k\to\infty} M_A = \lim_{k\to\infty} \frac{q_0 l^2}{2}\left(\frac{3l^3}{4\left(3\frac{EI}{k}+l^3\right)} - 1\right) = -\frac{q_0 l^2}{8}$$

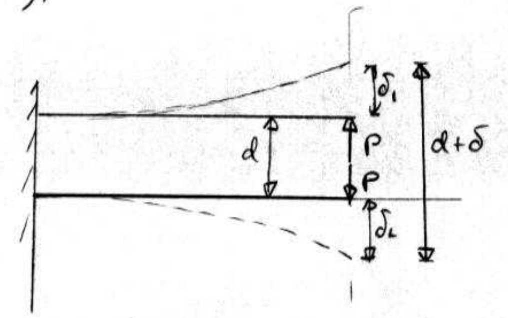
$$\lim_{k\to\infty} v = \lim_{k\to\infty} \frac{q_0}{48EI}\left(12l^2x^2 - 8lx^3 + 2x^4 - \frac{3q_0 l^4}{\left(\frac{3EI}{k}+l^3\right)}(3lx^2 - x^3)\right)$$

$$= \frac{q_0}{48EI}\left(12l^2x^2 - 8lx^3 + 2x^4 - 3q_0 l(3lx^2 - x^3)\right)$$

$$= \frac{q_0}{48EI}\left(3l^2x^2 - 5lx^3 + 2x^4\right)$$ Fr. 9a) opps. 3.

Oppg. 3





$$\left.\begin{aligned}\delta_1 &= -\frac{Pl^3}{3E_1I_1}\\ \delta_2 &= \frac{Pl^3}{3E_2I_2}\end{aligned}\right\} \quad \text{Fra oppg. 1.}$$

$$\delta = \delta_2 - \delta_1 = \frac{Pl^3}{3E_2I_2} + \frac{Pl^3}{3E_1I_1} = \frac{Pl^3}{3}\left(\frac{1}{E_2I_2} + \frac{1}{E_1I_1}\right)$$

$$\Rightarrow P = \frac{3\delta E_2I_2E_1I_1}{l^3(E_2I_2 + E_1I_1)}$$

$$I_1 = \frac{1}{12}bh_1^3 \quad , \quad I_2 = \frac{1}{12}bh_2^3$$

$$P = \frac{3\delta E_1 E_2 \cdot \frac{1}{12} b h_1^3 \, \frac{1}{12} b h_2^3}{\frac{l^3 b}{12}\left(E_2 h_2^3 + E_1 h_1^3\right)} = \frac{\delta E_1 E_2 b h_1^3 h_2^3}{4 l^3 \left(E_2 h_2^3 + E_1 h_1^3\right)}$$

Maksimal spenning oppstår ved innspenningen.

$$\sigma_m = \frac{M}{I_1} \cdot \frac{h_1}{2}$$

$$M = P \cdot l = \frac{\delta E_1 E_2 b h_1^3 h_2^3}{4 l^2 \left(E_2 h_2^3 + E_1 h_1^3\right)}$$

$$\sigma_m = \frac{\dfrac{\delta E_1 E_2 \cancel{b} \cancel{h_1^3} h_2^3}{4 l^2 \left(E_2 h_2^3 + E_1 h_1^3\right)} \cdot \dfrac{h_1}{2}}{\frac{1}{12} \cancel{b} \cancel{h_1^3}}$$

$$= \frac{3 \delta E_1 E_2 h_1 h_2^3}{2 l^2 \left(E_1 h_1^3 + E_2 h_2^3\right)}$$