UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MEK2500 — Solid mechanics

Day of examination: 04.12.2015

Examination hours: 14.30 – 18.30

This problem set consists of 4 pages.

Appendices: None

Permitted aids: Rottman's formulae and Approved Calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 (weight 20%)

This problem consists of 5 independent questions/tasks. Each question/task has a max score of 2 points. Explain your answers here and throughout.

a) (2 points) For a rigid body deformation f and its associated (Cauchy's infinitesimal) strain tensor ε, what is the norm of ε?

Answer: 0. The strain tensor vanishes for rigid body deformations.

b) (2 points) For a constant diagonal strain tensor $\varepsilon \in \mathbb{R}^{3,3}$, with elements d_1, d_2, d_3, find the largest principal strain.

Answer: Assume that $0 < d_3 \leq d_2 \leq d_1$ without loss of generality. Then the largest principal strain is d_1.

c) (2 points) Given a Cauchy's stress tensor σ, express the normal stress and shear stresses acting on a plane with normal n.

Answer: By definition, normal stress $s_n = \sigma n \cdot n$, shear stresses are $s_t = \sigma n \times n$.

d) (2 points) Give an example of a stress tensor that describes an isotropic pressure.

Answer: One example is $\sigma = -pI$ where p is a scalar field and I is the identity matrix.

e) (2 points) Define the yield strength of an elastic material.

Answer: The yield strength of an elastic material is the maximal tensile stress that the material can sustain before yielding, i.e. undergoing plastic (permanent) deformation.

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Problem 2 (weight 40%)

Assume that an elastic body B of unit volume occupies a domain $\Omega \subset \mathbb{R}^3$ with coordinates $x = (x_1, x_2, x_3)$. Assume that the body is deformed with a displacement u given by:

$$u(x) = (\kappa x_1, -\kappa x_2, 0) \tag{1}$$

for $\kappa > 0$.

a) (5 points) Compute Cauchy's infinitesimal strain tensor ε of the displacement u.

Answer: By definition

$$\nabla u = \begin{pmatrix} \kappa & 0 & 0 \\ 0 & -\kappa & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2}$$

and $\varepsilon = \nabla u$ since ∇u is symmetric.

b) (5 points) What is the change in volume of B induced by the displacement u?

Answer: The relative change in infinitesimal volumes is given by the divergence of the displacement.

$$\nabla \cdot u = \operatorname{tr} \varepsilon = \kappa - \kappa = 0 \tag{3}$$

The relative change in volume is zero, and thus all infinitesimal volumes remain constant, so does the total volume.

c) (5 points) (i) Compute the Cauchy stress tensor associated with this displacement u, assuming that the body is isotropic, homogeneous and linearly elastic with Lamé parameters μ and λ. (ii) Compute the resulting stress on planes orthogonal to the vector $t = (1, 1, 1)$.

Answer: Under these assumptions, Hooke's law for an isotropic material holds. Thus Cauchy's stress tensor is given by:

$$\sigma = 2\mu\varepsilon + \lambda \operatorname{tr} \varepsilon I \tag{4}$$

Since

$$\operatorname{tr} \varepsilon I = \nabla \cdot u = 0 \tag{5}$$

We have that

$$\sigma = 2\mu\varepsilon = 2\mu \begin{pmatrix} \kappa & 0 & 0 \\ 0 & -\kappa & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6}$$

The plane orthogonal to the vector t has normal direction $n = t/\|t\|$ where $\|t\| = \sqrt{3}$. The stress on this plane is given by

$$\sigma n = \frac{1}{\sqrt{3}} \sigma(1, 1, 1) = \frac{2\mu}{\sqrt{3}} (\kappa, -\kappa, 0) \tag{7}$$

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d) (5 points) Compute the principal stresses and principal directions of stress associated with this displacement.

Answer: The principal stresses and principal directions of stress are given by the eigenvalues and eigenvectors of the stress tensor, respectively. For a diagonal matrix, the eigenvalues are given by the diagonal elements. Thus the principal stresses are $\lambda_1 = 2\mu\kappa, \lambda_2 = -2\mu\kappa, \lambda_3 = 0.0$. The associated principal directions of strain are given by the standard Cartesian basis $v_1 = e_1, v_2 = e_2, v_3 = e_3$.

Problem 3 (weight 40%)

Consider a two-dimensional elastic body A occupying a domain $\Omega = [-1, 1] \times [-b, b] \subset \mathbb{R}^2$ with Cartesian coordinates $x = (x_1, x_2)$. Denote the boundary of Ω by $\partial\Omega$ and the outward normal vector (field) on the boundary $\partial\Omega$ by $n = n(x)$.

Assume a linear regime with no distinction between Eulerian and Lagrangian coordinates. Assume that the body A is subjected to a body force (field) $f = f(x)$ and a boundary stress (field) $g = g(x)$ acting on $\partial\Omega$.

Consider this setting for the questions below.

a) (5 points) State the mechanical equilibrium equation(s) for the stress tensor σ in terms of f and g.

Answer:

$$\begin{aligned} -\nabla \cdot \sigma &= f \quad \text{in } \Omega, \\ \sigma n &= g \quad \text{on } \partial\Omega. \end{aligned}$$

b) Assume that the body A is shear stress free (i.e. $\sigma_{12} = 0$). Let $f = (0, 0)$. Let $g = (g_1(x), 0)$ where

$$g_1(x) = g_1(x_1, x_2) = x_1(x_2 - b)(x_2 + b). \tag{8}$$

Compute the stress tensor σ.

Answer: By the assumption of shear stress free, we know that $\sigma_{12} = 0$. Thus, by symmetry:

$$\sigma(x_1, x_2) = \begin{pmatrix} \sigma_1(x_1, x_2) & 0 \\ 0 & \sigma_2(x_1, x_2) \end{pmatrix} \tag{9}$$

With this and since $f = (0, 0)$, we know that

$$\begin{aligned} (\nabla \cdot \sigma)_1 = \partial_{x_1}\sigma_1(x_1, x_2) = 0 \\ (\nabla \cdot \sigma)_2 = \partial_{x_2}\sigma_2(x_1, x_2) = 0 \end{aligned}$$

Thus $\sigma_1(x_1, x_2) = \sigma_1(x_2)$ and $\sigma_2(x_1, x_2) = \sigma_2(x_1)$.

On the top boundary, where $x_2 = b$, $n_T = (0, 1)$ and so $\sigma n_T = (0, \sigma_2(x_1))$. For this: $(0, \sigma_2(x_1)) = (g_1(x), 0)$ to hold for all x_1, $\sigma_2(x_1) = 0$. This is compatible with the bottom boundary and allows

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for the boundary condition to hold there. Since $g_1(x_1, \pm b) = 0$, this is ok.

On the left boundary, where $x_1 = -1$, $n_L = (-1, 0)$ and so $\sigma n_L = (-\sigma_1(x_2), 0)$. For this: $(-\sigma_1(x_2), 0) = (g_1(-1, x_2), 0)$ to hold, $\sigma_1(x_2) = -g_1(-1.0, x_2)$ for all x_2. Thus

$$\sigma_1(x_2) = -g_1(-1.0, x_2) = (x_2 - b)(x_2 + b) \tag{10}$$

Since, $g_1(-1.0, x_2) = -g_1(1.0, x_2)$ for all x_2, this is compatible with the right boundary.

In conclusion,

$$\sigma(x_1, x_2) = \begin{pmatrix} (x_2 - b)(x_2 + b) & 0 \\ 0 & 0 \end{pmatrix} \tag{11}$$

c) (5 points) Using von Mises yield criterion, where is the body most likely to yield?

Answer: The trace of the stress tensor is:

$$\operatorname{tr} \sigma = (x_2 - b)(x_2 + b) \tag{12}$$

Thus 11-component of the deviatoric stress tensor is the only non-zero and it reads:

$$\sigma_{\text{dev},11} = \sigma_{11} - \frac{1}{3} \operatorname{tr} \sigma = \frac{2}{3}(x_2 - b)(x_2 + b) \tag{13}$$

Thus

$$\sigma_{\text{vM}}^2 = \frac{3}{2}\frac{2}{3}\frac{2}{3}((x_2 - b)(x_2 + b))^2 \tag{14}$$

and so

$$\sigma_{\text{vM}} = \frac{1}{3}|(x_2 - b)(x_2 + b)| \tag{15}$$

According to von Mises yield criterion, the body is most likely to yield where the von Mises stresses are largest, which is at $x_2 = 0$.

d) (5 points) Now, instead let $f = (F, 0)$. Assume that the stress in the body is still described by the stress tensor σ in 3b). Compute the resulting accelleration $a = a(x)$ of the body.

Answer:

The dynamic elasticity equations read:

$$\rho a = \nabla \cdot \sigma + f \tag{16}$$

For the stress tensor from 3b), $\nabla \cdot \sigma = 0$. Thus $a = f/\rho = (F/\rho, 0)$.