Note on isotropy for MEK3220

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Isotropy means that a property is independent of direction, which implies that the property can be quantified by the same physical value(s) independent of rotations of the frame of reference. An isotropic tensor has the same components in all rotated coordinate systems.

A tensor of rank 0, which is simply a scalar (e.g. temperature, salinity, etc.), does not depend on the orientation of the coordinate axes at all. Thus the scalar is isotropic.

In order to understand isotropy for vectors and tensors of higher rank, we first need to consider the effect of rotating the frame of reference.

Let $\{\boldsymbol{i}_j\}$ and $\{\boldsymbol{i}'_j\}$, with $j = 1, 2, 3$, be two orthonormal sets of basis vectors that each span three-dimensional Eucledian space. Thus $\boldsymbol{i}_j \cdot \boldsymbol{i}_k = \delta_{jk}$ and $\boldsymbol{i}'_j \cdot \boldsymbol{i}'_k = \delta_{jk}$.

Let us consider how the basis vectors transform between the primed and unprimed systems. Suppose

$$\boldsymbol{i}'_j = \ell_{jk}\boldsymbol{i}_k \qquad \text{and} \qquad \boldsymbol{i}_j = \ell'_{jk}\boldsymbol{i}'_k \tag{1}$$

Upon taking the scalar product of the first equation with $\boldsymbol{i}_l$ and of the second equation with $\boldsymbol{i}'_k$, and using the fact that the basis vectors are orthonormal, we get

$$\ell_{jl} = \boldsymbol{i}'_j \cdot \boldsymbol{i}_l \qquad \text{and} \qquad \ell'_{jl} = \boldsymbol{i}_j \cdot \boldsymbol{i}'_l \tag{2}$$

Thus these are the cosines of the angles between each pair of basis vectors from the two reference systems.

Consider the obvious identities

$$\ell_{jl} = \ell'_{lj} \tag{3}$$

and

$$\boldsymbol{i}'_j = \ell_{jk}\boldsymbol{i}_k = \ell_{jk}\ell'_{kl}\boldsymbol{i}'_l = \ell_{jk}\ell_{lk}\boldsymbol{i}'_l \qquad \text{and} \qquad \boldsymbol{i}_j = \ell'_{jk}\boldsymbol{i}'_k = \ell'_{jk}\ell_{kl}\boldsymbol{i}_l = \ell_{kj}\ell_{kl}\boldsymbol{i}_l \tag{4}$$

thus we get the two important relationships

$$\ell_{jk}\ell_{lk} = \delta_{jl} \qquad \text{and} \qquad \ell_{kj}\ell_{kl} = \delta_{jl} \tag{5}$$

This can also be written in matrix form as

$$L^T L = L L^T = I \tag{6}$$

where I is the identity matrix, and L is the matrix represented by ℓ_{jk}, and L^T is the transpose of L. A real matrix that satisfies this relationship is called an orthogonal matrix.

Consider a representation of a vector $\boldsymbol{v}$ in the two frames of reference. We have

$$\boldsymbol{v} = v_j\boldsymbol{i}_j = v_j\ell_{kj}\boldsymbol{i}'_k = v'_k\boldsymbol{i}'_k \qquad \text{and} \qquad \boldsymbol{v} = v'_j\boldsymbol{i}'_j = v'_j\ell_{jk}\boldsymbol{i}_k = v_k\boldsymbol{i}_k \tag{7}$$

thus the vector components transform in the same way as the basis vectors

$$v'_k = \ell_{kj} v_j \qquad \text{and} \qquad v_k = \ell_{jk} v'_j \tag{8}$$

Let us consider the existence of an isotropic tensor of rank 1, or an isotropic vector. If such a vector exists, then the components of the vector must be independent of the reference system. Thus we get the condition

$$v'_j = v_j = \ell_{jk} v_k \tag{9}$$

First consider a rotation by $\pi/2$ around the third axis

$$L = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{10}$$

which requires $v_1 = v_2 = 0$. Then consider a similar rotation by $\pi/2$ around the first axis, which would require $v_2 = v_3 = 0$. Thus we conclude that there is no nontrivial isotropic vector.

Before considering isotropic tensors of higher rank, we may take advantage of this moment to define, for the first time in this course, what we really mean by a tensor. (*Here we limit attention to so-called covariant tensors only.*) A tensor of rank n is a quantity $A_{j_1 j_2 \ldots j_n}$ that transforms according to the following law

$$A'_{j_1 j_2 \ldots j_n} = \ell_{j_1 k_1} \ell_{j_2 k_2} \ldots \ell_{j_n k_n} A_{k_1 k_2 \ldots k_n} \tag{11}$$

where ℓ_{jk} are the directional cosines as defined above. Thus the defining property of a tensor is that each index transforms as if it represented a vector.

Let us consider isotropic tensors of rank 2. Here we shall limit our effort to show that the Kronecker delta δ_{ij} is an isotropic tensor of rank 2. We need to show that

$$\delta_{ij} = \ell_{ik} \ell_{jl} \delta_{kl} \tag{12}$$

Using the properties of the Kronecker delta, and the properties of orthogonal matrices, the right-hand side becomes

$$\ell_{ik} \ell_{jl} \delta_{kl} = \ell_{ik} \ell_{jk} = \delta_{ij} \tag{13}$$

Thus we have shown that the Kronecker delta is an isotropic tensor. It can furthermore be shown that this is the only isotropic tensor of rank 2. Demonstration of uniqueness is left as an exercise.

For our purposes we do not need to be concerned about the fact that there is one isotropic tensor of rank 3.

In the following we shall give a demonstration of three isotropic tensors of rank 4. First consider the rank 4 tensor $A_{ijkl} = \delta_{ij} \delta_{kl}$. We need to show that

$$\delta_{ij} \delta_{kl} = \ell_{ip} \ell_{jq} \ell_{kr} \ell_{ls} \delta_{pq} \delta_{rs} \tag{14}$$

Using the properties of the Kronecker delta twice, and the properties of orthogonal matrices twice, the right-hand side becomes

$$\ell_{ip} \ell_{jq} \ell_{kr} \ell_{ls} \delta_{pq} \delta_{rs} = \ell_{ip} \ell_{jp} \ell_{kr} \ell_{lr} = \delta_{ij} \delta_{kl} \tag{15}$$

which is what we needed to show. Second consider another rank 4 tensor $A_{ijkl} = \delta_{ik}\delta_{jl}$. We show that

$$\ell_{ip}\ell_{jq}\ell_{kr}\ell_{ls}\delta_{pr}\delta_{qs} = \ell_{ip}\ell_{jq}\ell_{kp}\ell_{lq} = \delta_{ik}\delta_{jl} \tag{16}$$

Third consider yet another rank 4 tensor $A_{ijkl} = \delta_{il}\delta_{jk}$. We show that

$$\ell_{ip}\ell_{jq}\ell_{kr}\ell_{ls}\delta_{ps}\delta_{qr} = \ell_{ip}\ell_{jq}\ell_{kq}\ell_{lp} = \delta_{il}\delta_{jk} \tag{17}$$

It can furthermore be shown that any isotropic tensor of rank 4 can be expressed as a linear combination of these three tensors, but a demonstration of this fact is left as an exercise.

To summarize, we are in the process of establishing a relationship between tensor rank and number of independent isotropic tensors

tensor rank	0	1	2	3	4	...
number of isotropic tensors	1	0	1	1	3	...

A commonly used representation of the most general isotropic tensor of rank 4 is

$$T_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \nu(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \tag{18}$$

Let us apply the concept of isotropy to Hooke's law of elasticity. Hooke's law of elasticity is the fundamental assumption that the stress is a linear function of the strain

$$P_{ij} = A_{ijkl}\epsilon_{kl} \tag{19}$$

Now if we assume that the material is isotropic, we may let the tensor A_{ijkl} in Hooke's law be given by the tensor T_{ijkl} discussed above.

$$\begin{aligned} P_{ij} &= \left[\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \nu(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\right]\epsilon_{kl} && (20) \\ &= \lambda\delta_{ij}\epsilon_{kk} + \mu(\epsilon_{ij} + \epsilon_{ji}) + \nu(\epsilon_{ij} - \epsilon_{ji}) && (21) \end{aligned}$$

Keeping in mind that the deformation tensor ϵ_{kl} is symmetric, we see that the stress–strain relationship is simplified to

$$P_{ij} = \lambda\delta_{ij}\epsilon_{kk} + 2\mu\epsilon_{ij} \tag{22}$$

We may ask what would have happened if we had used a non-symmetric deformation tensor instead of the symmetric strain tensor in Hooke's law. In that case, the stress contribution proportional to ν would be anti-symmetric, in contradiction to the second stress principle of Cauchy which requires the stress tensor to be symmetric. This may help to motivate why only the two elastic coefficients λ and μ are relevant for an isotropic medium.

Similarly, we can apply the concept of isotropy to a Newtonian fluid. For a Newtonian fluid the fundamental assumption is made that the viscous stress is a linear function of the rate of strain

$$\sigma_{ij} = A_{ijkl}\dot{\epsilon}_{kl} \tag{23}$$

while the total stress is the linear superposition of isotropic pressure p and viscous stress

$$P_{ij} = -p\delta_{ij} + \sigma_{ij} = -p\delta_{ij} + A_{ijkl}\dot{\epsilon}_{kl} \tag{24}$$

If we require tha the medium is isotropic, and by an argument similar to that of isotropic elasticity, and keeping in mind that the rate of strain tensor is also symmetric, we end up with the expression

$$P_{ij} = -p\delta_{ij} + \lambda\delta_{ij}\dot{\epsilon}_{kk} + 2\mu\dot{\epsilon}_{ij} \tag{25}$$

Exercise: Show that the principal stress directions are equal to the principal strain directions in an isotropic linear elastic medium.

Exercise: Show that the principal stress directions are equal to the principal rate-of-strain directions in an isotropic Newtonian fluid.