MEK3220, selected formulae

Notation

β and $\mathbf{A}$ are used to denote a general scalar field and a vector field, respectively.

Cylinder coordinates

Transformation $x = r\cos\theta$, $y = r\sin\theta$, $\mathbf{i}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$, $\mathbf{i}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$.
The gradient operator

$$\nabla = \mathbf{i}_r \frac{\partial}{\partial r} + \frac{1}{r}\mathbf{i}_\theta \frac{\partial}{\partial \theta} + \mathbf{k}\frac{\partial}{\partial z}$$

Differentiation:

$$\nabla^2 \beta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \beta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \beta}{\partial \theta^2} + \frac{\partial^2 \beta}{\partial z^2}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_r) + \frac{\partial A_\theta}{\partial \theta}\right) + \frac{\partial A_z}{\partial z}$$

$$\begin{aligned}\nabla \mathbf{A} = \; & \tfrac{\partial A_r}{\partial r}\mathbf{i}_r\mathbf{i}_r + \tfrac{\partial A_\theta}{\partial r}\mathbf{i}_r\mathbf{i}_\theta + \tfrac{\partial A_z}{\partial r}\mathbf{i}_r\mathbf{k} \\ & + \tfrac{1}{r}\left(\tfrac{\partial A_r}{\partial \theta} - A_\theta\right)\mathbf{i}_\theta\mathbf{i}_r + \tfrac{1}{r}\left(A_r + \tfrac{\partial A_\theta}{\partial \theta}\right)\mathbf{i}_\theta\mathbf{i}_\theta + \tfrac{1}{r}\tfrac{\partial A_z}{\partial \theta}\mathbf{i}_\theta\mathbf{k} \\ & + \tfrac{\partial A_r}{\partial z}\mathbf{k}\mathbf{i}_r + \tfrac{\partial A_\theta}{\partial z}\mathbf{k}\mathbf{i}_\theta + \tfrac{\partial A_z}{\partial z}\mathbf{k}\mathbf{k}\end{aligned}$$

$$\nabla^2 \mathbf{A} = \left(\nabla^2 A_r - \frac{A_r}{r^2} - \frac{2}{r^2}\frac{\partial A_\theta}{\partial \theta}\right)\mathbf{i}_r + \left(\nabla^2 A_\theta - \frac{A_\theta}{r^2} + \frac{2}{r^2}\frac{\partial A_r}{\partial \theta}\right)\mathbf{i}_\theta + \left(\nabla^2 A_z\right)\mathbf{k}.$$

where $\mathbf{A} = A_r\mathbf{i}_r + A_\theta\mathbf{i}_\theta + A_z\mathbf{k}$.

Spherical coordinates

Transformation $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$,
$\mathbf{i}_r = \sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k}$,
$\mathbf{i}_\phi = -\sin\phi\mathbf{i} + \cos\phi\mathbf{j}$,
$\mathbf{i}_\theta = \cos\theta\cos\phi\mathbf{i} + \cos\theta\sin\phi\mathbf{j} - \sin\theta\mathbf{k}$.
Differentiation: ($\mathbf{A} = A_r\mathbf{i}_r + A_\phi\mathbf{i}_\phi + A_\theta\mathbf{i}_\theta$)

$$\nabla = \mathbf{i}_r\frac{\partial}{\partial r} + \frac{1}{r\sin\theta}\mathbf{i}_\phi\frac{\partial}{\partial \phi} + \frac{1}{r}\mathbf{i}_\theta\frac{\partial}{\partial \theta}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta)$$

Gauss' theorem

σ is the closed surface of the finite volume τ.

$$\int_\sigma \mathbf{n}\cdot\mathbf{A}d\sigma = \int_\tau \nabla\cdot\mathbf{A}d\tau, \quad \int_\sigma \mathbf{n}\cdot\mathcal{P}d\sigma = \int_\tau \nabla\cdot\mathcal{P}d\tau, \quad \int_\sigma \mathbf{n}\beta d\sigma = \int_\tau \nabla\beta d\tau$$

The heat equation

Specific density of thermal energy for incompressible media: $E(T)$.
Heat capacity: $c = \frac{\partial E}{\partial T}$.
Fouriers law for heat flux density $q_x = -k\nabla T$ (conduction).
The heat equation:

$$\frac{\mathbf{D}T}{\mathbf{d}t} = \kappa \nabla^2 T + \frac{\Delta}{\rho c}, \quad \text{where} \quad \kappa = \frac{k}{\rho c}.$$

In Cartesian coordinates the dissipation for an incompressible fluid is (sum over i and j)

$$\Delta = 2\mu \dot{\epsilon}_{ij} \dot{\epsilon}_{ij}.$$